# Supersymmetric backgrounds of M-theory and AdS4/CFT3 correspondence

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# Supersymmetric backgrounds of M-theory and $AdS_4/CFT_3$ correspondence

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## Abstract

We analyse the most general $\mathcal{N} = 2$ supersymmetric solutions of $d = 11$ supergravity consisting of a warped product of four-dimensional anti-de-Sitter space with a seven-dimensional Riemannian manifold $Y_7$. We show that the necessary and sufficient conditions for supersymmetry can be phrased in terms of a local $SU(2)$-structure on $Y_7$. Solutions with non-zero M2-brane charge also admit a canonical contact structure, in terms of which many physical quantities can be expressed, including the free energy and the scaling dimensions of operators dual to supersymmetric wrapped M5-branes. We show that a special class of solutions is singled out by imposing an additional symmetry, for which the problem reduces to solving a second order non-linear ordinary differential equation. As well as recovering a known class of solutions, that includes the IR fixed point of a mass deformation of the ABJM theory, we also find new solutions which are dual to cubic deformations. In particular, we find a new supersymmetric warped $\mathrm{AdS}_4 \times S^7$ solution with non-trivial four-form flux. Furthermore, we study supersymmetric asymptotically locally $\mathrm{AdS}_4$ solutions of $\mathcal{N} = 2$ gauged supergravity which via the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence are dual to supersymmetric gauge theories on deformed 3-spheres with $SU(2) \times U(1)$ symmetry and a non-trivial background gauge field. These solutions lift to solutions of M-theory and we show that the gravitational free energy agrees with the large $N$ limit of the dual field theory free energy, obtained from the localized partition function of a class of $\mathcal{N} = 2$ Chern-Simons-matter theories. In this context, we present a complete class of supersymmetric backgrounds of $\mathcal{N} = 2$ gauged supergravity whose conformal boundary is a biaxially squashed Lens space $S^3/\mathbb{Z}_p$. Generically we find that the latter admits Taub-NUT-AdS fillings, with topology $\mathbb{R}^4/\mathbb{Z}_p$, as well as smooth Taub-Bolt-AdS fillings with non-trivial topology.

# 1 Introduction

The preeminent theoretical framework for the study of the fundamental structures of nature is string theory. String theory provides a consistent theory of quantum gravity and a unified description of the fundamental forces. It has led to new insights into the character of quantum field theories, which underlie the description of phenomena of high energy and condensed matter physics. An instance of these insights is the duality between gravitational theories and gauge field theories (gauge/gravity duality) which has been extensively studied in recent years and produced fascinating results in various disciplines of physics and mathematics.

### String Theory

String theory was formulated in the late 1960's as a theory of hadronic interactions. It was abandoned in favor of Quantum Chromodynamics but it was soon realized that it could serve as a theory of quantum gravity with the potential of unifying all forces of nature.

Contrary to the quantum field theories of elementary particles which treat the latter as mathematical points in spacetime, the fundamental objects in string theory are one-dimensional i.e. strings, open or closed. As a string evolves in time it sweeps a two-dimensional surface known as the worldsheet of the string. The string can oscillate in various modes and at large distances compared to the characteristic string length $\ell_s$, each mode gives rise to a different species of particle. In the spectrum of string theory there is a massless particle with spin two whose interactions are that of a graviton. Thus, although modified at high energies, general relativity is naturally incorporated in string theory. In addition, Yang-Mills theories which are the cornerstone of the Standard Model of high-energy physics, also arise in string theory, rendering the latter a framework for the unification of all forces.

In perturbative string theory the interactions are described by smooth worldsheets of various topologies and hence the interaction does not take place on a singular point in spacetime. This is in contrast with conventional particle physics where the interactions are located on singular points of the interacting particles' worldlines. As a consequence the structure of the interactions in string theory is uniquely determined by the free theory rather than being arbitrary as in quantum field theories. In addition, because the worldsheets are smooth the amplitudes in string perturbation theory are free of ultraviolet divergences.

The action of an supersymmetric string (superstring) reads

$$S = \frac{1}{4\pi\ell_s^2}\int \mathrm{d}^2\sigma\sqrt{g}\left(g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi\right) \tag{1.1}$$

Here $g_{ab}$ is the metric on the worldsheet and $\gamma^\mu$ Dirac matrices in two dimensions. $X^\mu$ are worldsheet scalars describing the embedding of the string in spacetime, and $\psi$ are worldsheet fermions. Depending on the boundary conditions of the fermions there are two sectors for the open supestring. The Neveu-Schwarz (NS) sector corresponding to anti-periodic boundary conditions and the Ramond (R) sector corresponding to periodic boundary conditions. The massless spectrum consists of a vector and Majorana-Weyl fermion. The closed string spectrum is obtained by tensoring right-moving and left-moving modes each of which are similar to the open string modes. There are thus four sectors for the closed string states: NS⊗NS, NS⊗R, R⊗NS and R⊗R. Spacetime bosons come from the sectors where the boundary conditions for the fermions are the same for right-moving and left-moving modes i.e. the NS⊗NS and R⊗R sectors while spacetime fermions come from the rest of the sectors. The massless spectrum of the NS⊗NS sector contains the graviton, a two form $B_{\mu\nu}$ and a scalar $\phi$ the dilaton. The R⊗R sector contains antisymmetric tensor fields or $p$-forms with various number of indices.

There are five superstring theories depending on the number of supercharges, their chirality and the inclusion or not of open strings:

- type IIA: $16+16$ supercharges of opposite chirality; the theory contains oriented closed strings.
- type IIB: $16+16$ supercharges of the same chirality; the theory contains oriented closed strings
- type I: 16 supercharges; the theory contains unoriented closed and open strings; $SO(32)$ gauge group.
- HO heterotic: 16 supercharges; the theory contains closed oriented strings; hybrid between superstring and bosonic string; $SO(32)$ gauge group
- HE heterotic: 16 supercharges; the theory contains closed oriented strings; hybrid between superstring and bosonic string; $E_8\times E_8$ gauge group

Since string theory contains massless particles separated by a large mass gap, inversely proportional to the string length $\ell_s$, from the massive states of the spectrum it is natural to study the effective low-energy theories. These are typically supergravity theories and in the case of open strings (super) Yang-Mills theories. The massless spectrum of the five superstring theories is summarized in table 1.1.

Although perturbatively it appears that there are five distinct superstring theories, these are connected via a web of dualities i.e. the five superstring theories are equivalent

| | Neveu-Schwarz | Ramond |
|---|---|---|
| type IIA | $g_{\mu\nu}, B_{\mu\nu}, \phi$ | $A_\mu, A_{\mu_1\mu_2\mu_3}$ |
| type IIB | $g_{\mu\nu}, B_{\mu\nu}, \phi$ | $A, A_{\mu\nu}, A_{\mu_1\ldots\mu_4}$ |
| type I | $g, \phi$ | $A_{\mu\nu}$ |
| HO/HE hetrotic | $g_{\mu\nu}, B_{\mu\nu}, \phi$ | - |

Table 1.1: The massless spectrum of superstring theories

or in other words there is a unique theory underlying them. One type of duality is called S-duality. Two string theories are related via S-duality if one of them evaluated at strong coupling is equivalent to the other one evaluated at weak coupling. S-duality connects HO superstring theory to type I whereas type IIB is selfdual.

Another duality is T-duality which relates different compactifications of different theories. Let theory A have a compact dimension that is a circle of radius $R_A$ and theory B a compact dimension of radius $R_B$. These theories are equivalent via T-duality if

$$R_A R_B = \ell_s^2 \tag{1.2}$$

More complicated examples involve compact spaces such as tori and Calabi-Yau manifolds. T-duality relates IIA and IIB theories as well as HE and HO theories.

Finally, there is an eleven dimensional non-perturbative extension of the superstring theories known as M-theory. The low energy dynamics of M-theory is captured by eleven-dimensional supergravity which consists of the graviton and a 3-form potential. M-theory arises as the strong coupling limit of type IIA superstring theory; type IIA theory has a circular eleventh dimension which decompactifies in the strong coupling limit yielding M-theory. The HE heterotic theory is also connected to M-theory in a similar way.

The web of dualities is summarized in figure 1.1.

Another non-perturbative aspect of string theory are D-branes. D$p$-branes are spatially $p$-dimensional objects which arise as solitons in string theory. They are extended surfaces in spacetime where strings can end; D stands for Dirichlet boundary conditions. Their tension or energy density diverges as the string coupling $g_s$ goes to zero and hence they are absent from string perturbation theory. Another characteristic is that they carry a charge that couples to the $p$-form Ramond-Ramond fields. In the low energy approximation they appear as extremal $p$-brane solutions of supergravity theories.

The end of an open string carries a charge and on the worldvolume of the D-brane there is an abelian gauge field which carries the associated flux. On a stack of $n$ coinci-

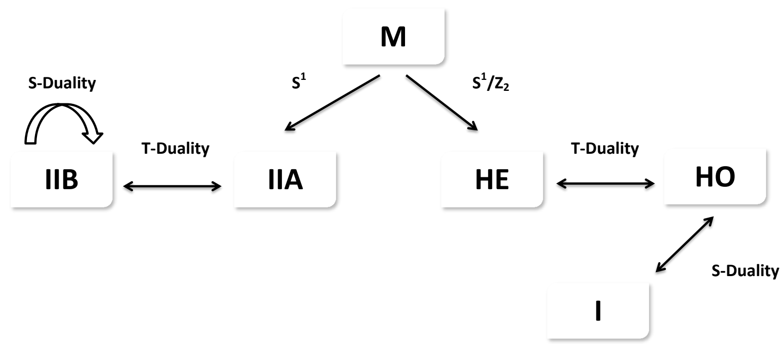


Figure 1.1: The web of dualities relating the superstring theories and M-theory

dent D$p$-branes the associated theory at low energies, living on the $p+1$ worldvolume, is a super Yang-Mills theory with gauge group $U(n)$. There are $n^2$ gauge bosons $A_\mu^{ij}$ which arise from open strings whose ends lie on the $i$th and $j$th D-brane respectively. The mass of the gauge bosons is proportional to the separation of the $i$th and $j$th D-brane. Hence when the D-branes are separated the symmetry is broken down to $U(1)^n$.

**Supersymmetry**

Supersymmetry is a putative symmetry of nature which has played a prominent role in recent developments in high energy physics. It is an extension of the Poincaré symmetry of spacetime which relates bosons, particles with integer spin, and fermions, particles with half-integer spin. A central motivation for supersymmetry is the resolution of the hierarchy problem of the Standard Model of particle physics. The Higgs boson mass in the Standard Model is subject to quadratically divergent quantum corrections which drive it close to the Planck mass. Supersymmetry alleviates the divergent quantum corrections by introducing cancellations between fermionic and bosonic Higgs interactions. Supersymmetry also allows for gauge coupling unification and provides candidate dark matter particles. Finally, supersymmetry is a key ingredient of string theory as it removes the tachyon that appears in the spectrum of the bosonic string theory.

Requiring supersymmetry to be a local symmetry, general relativity appears as a consequence and the resulting theory is a supergravity theory. Supergravity is appealing as a physical theory, as supersymmetry imposes stringent constraints on its dynamics and field content, giving rise to rich mathematical structures. Supersymmetry has the property of alleviating the divergent ultraviolet behaviour of quantum field theories and thereupon, supergravity was originally conceived as a fundamental theory, capable of eliminating the non-renormalizable divergences that appear in the construction of a quantum theory of gravity. The current view is that although local supersymmetry improves the high-energy behaviour of quantum gravity, supergravity is an effective rather than a fundamental theory of nature. As mentioned earlier the massless sector of the spectrum of superstring theories is described by supergravity and thus by studying the behaviour of classical supergravity solutions, one retrieves valuable information about the low-energy dynamics of superstring theories. In addition, many results established at the supergravity level, such as dualities connecting different coupling regimes of various supergravity theories, can be elevated to the superstring level.

### Supersymmetric compactifications

Superstring theories and M-theory reside in ten and eleven dimensions respectively. In pursuit of realistic vacua one thus considers compactifications of superstring theories or M-theory on six or seven-dimensional compact manifolds. Usually one works in the low-energy limit i.e. supergravity theories. Supersymmetry is a desired symmetry of such backgrounds as the conditions for supersymmetry imply the equations of motion[1] and are easier to solve compared to the latter. In addition supersymmetry imposes constrains on the geometry of the compact internal manifolds.

Requirement of supersymmetry is equivalent to the existence of a spinorial parameter $\epsilon$ such that the variations $\delta_\epsilon$ of the supergravity fields under supersymmetry transformations vanish. Typically the supersymmetric conditions come from the variation of the fermionic fields (which are set to zero) and are differential or algebraic.

Extensively studied compactifications are those in which all the $p$-form fields or fluxes are set to zero. The requirement for supesymmetry in this case is equivalent to the existence of a parallel spinor $\epsilon$ on the internal manifold

$$\nabla_\mu \epsilon \;=\; 0\;. \tag{1.3}$$

There is a one-to-one correspondence between parallel spinors and spinors invariant under the holonomy group $\mathrm{Hol}(\nabla)$ so that the internal manifolds admitting parallel spinors are special holonomy manifolds [1]. Typical examples are manifolds of $SU(3)$ holonomy or Calabi-Yau manifolds in six dimensions and manifolds of $G_2$ holonomy in seven dimensions.

More general compactifications are flux compactifications where the $p$-form fields are turned on. In that case the internal space is no longer a manifold of special holonomy; there is a back-reaction of the fluxes on the geometry. An organizational principle of this back-reaction is that of $G$-structures.

A $G$-structure on an $n$-dimensional manifold $M$, for a given structure group $G$, is a $G$-subbundle of the tangent frame bundle of $M$. The structure group of the latter is in general $GL(n,\mathbb{R})$ and tensors on $M$ transform in a representation of $GL(n,\mathbb{R})$. If a $G$-structure reduces $GL(n,\mathbb{R})$ to $G$, then singlets may occur in the decomposition of $GL(n,\mathbb{R})$ tensors into irreducible representations of $G$ and these singlets can be used as an alternative definition of the $G$-structure. In the case of supersymmetric backgrounds

[1] The Bianchi identities for the Ramond-Ramond fields need to be imposed independently.

the existence of globally defined spinors on a $n$-dimensional spin manifold reduces the Spin($n$) structure group to $G$, where $G \subset$ Spin($n$) is the stability subgroup of the spinors. $G$-invariant tensors can then be constructed as bilinears of the spinors.

In the presence of fluxes (1.3) is modified as

$$\nabla^t_\mu \epsilon \;=\; 0 \tag{1.4}$$

where $\nabla^t$ is a generalized connection comprising the Levi-Civita connection as well flux contributions. The intrinsic torsion $\nabla^t - \nabla$ can then be decomposed into $G$-modules i.e. torsion classes which fully characterize the $G$-structure.

As an example we consider the case of an $SU(3)$ structure in six dimensions. It is defined by the existence of a Weyl spinor $\epsilon$, or equivalently by a real two-form $\omega$ and a complex three-form $\Omega$ constructed as spinor bilinears, and satisfying

$$\omega \wedge \Omega \;=\; 0\;, \qquad \Omega \wedge \bar{\Omega} \;=\; \frac{8\mathrm{i}}{3!}\omega \wedge \omega \wedge \omega \;=\; 8\mathrm{i}\mathrm{vol}_6 \tag{1.5}$$

In the case that $\epsilon$ is parallel

$$\mathrm{d}\omega \;=\; \mathrm{d}\Omega \;=\; 0 \tag{1.6}$$

and the six-manifold is Calabi-Yau manifold of $SU(3)$ holonomy. $\omega$ and $\Omega$ are then the Kälher and holomorphic (3,0) form respectively. More generally $\epsilon$ is parallel with respect to a generalized connection. In that case $\omega$ and $\Omega$ are no longer closed but satisfy

$$\begin{aligned}
\mathrm{d}\omega &= -\frac{3}{2}\mathrm{Im}\,(\bar{w}_1\Omega) + w_4 \wedge \omega + w_3\;, \\
\mathrm{d}\Omega &= w_1 \omega \wedge \omega + w_2 \wedge \omega + \bar{w}_5 \wedge \Omega
\end{aligned} \tag{1.7}$$

where $w_i$ are the torsion classes. The presence or not of a torsion class characterizes the six-manifold and examples are given in table 1.2 [2].

### Gauge/Gravity duality

The connection between certain limits of gauge field theories and theories that describe string excitations has a long history [3] but it was concretely realized by the Anti-deSitter/Conformal Field Theory (AdS/CFT) correspondence [4, 5, 6]. The AdS/CFT correspondence emerged from the study of dynamics of D-branes in string theory and provides an explicit example of a duality between a gauge field theory without gravitational degrees of freedom and a theory of gravity, string theory, on certain backgrounds. Furthermore, the gravitational theory resides in a higher dimensional spacetime and thus the AdS/CFT correspondence realizes a proposed property of quantum gravity, the holographic principle, which states that the degrees of freedom of a region of space are encoded on its boundary. The AdS/CFT correspondence and its generalizations are a

| torsion classes | manifold |
|---|---|
| $w_1 = w_2 = 0$ | complex |
| $w_1 = w_3 = w_4 = 0$ | symplectic |
| $w_1 = w_2 = w_3 = w_4 = 0$ | Kälher |
| $w_1 = w_2 = w_4 = w_5 = 0$ | special hermitean |
| $\mathrm{Im}\, w_1 = \mathrm{Im}\, w_2 = w_4 = w_5 = 0$ | half-flat |
| $w_1 = w_2 = w_3 = w_4 = w_5 = 0$ | Calabi-Yau |

Table 1.2: Examples of six-dimensional geometries with vanishing $SU(3)$ torsion classes

broad and active field of research. It provides a framework for studying the strongly coupled limit of quantum field theories, which is typically inaccessible via the conventional methods of field theory and which becomes important in the study of the low-energy dynamics of Quantum Chromodynamics and in certain condensed matter systems. In addition, it represents a complete formulation of a quantum theory of gravity, in terms of a quantum field theory.

The prototypical example of the AdS/CFT correspondence involves type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ on one side and four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(n)$ gauge group on the other side.

Consider a stack of $N$ coincident D3 branes in type IIB superstring theory. There are two complementary points of view of this system. One point of view depicts the D3 branes as the host of the open-string excitations and involves perturbation theory around flat spacetime. This picture is valid for $g_s N \ll 1$. The low energy limit consists of massless open and closed strings. The spectrum of massless open strings is that of $\mathcal{N} = 4$ super-Yang-Mills with $SU(N)$ gauge group living on the worldvolume of the D3 branes whereas in the bulk the massless spectrum of the closed strings is that of IIB supergravity. At low energy the open strings remain interacting, as the gauge coupling is dimensionless in $3+1$ dimensions while the closed strings have irrelevant interactions and decouple. The second point of view depicts the D3 branes as a source of closed strings and involves perturbation theory around the black 3-brane background. This picture is valid for $g_s N \gg 1$. The black 3-brane background for $N$ units of flux is

$$\begin{aligned} \mathrm{d}s^2 &= H^{-1/2}\mathrm{d}x_4^2 + H^{1/2}(r)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2) \ , \\ F_5 &= 2\pi N(1+\star)\frac{\mathrm{vol}(S^5)}{\pi^3} \ , \end{aligned} \tag{1.8}$$

| $\mathcal{N}=4$ $SU(N)$ super Yang-Mills | type IIB string theory on $\mathrm{AdS}_5\times S^5$ |
|---|---|
| all $N$ and $g_{YM}$ | full quantum theory |
| fixed $g_{YM}^2 N$ and $N\to\infty$ | classical limit |
| large $g_{YM}^2 N$ and $N\to\infty$ | classical supergravity limit |

Table 1.3: Limits of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

where $\mathrm{d}\Omega_5^2$ is the metric of a five-sphere of radius one, and

$$H(r) \;=\; 1+\frac{L^4}{r^4}\,, \qquad L^4 \;=\; 4\pi g_s N \ell_s^4\,. \tag{1.9}$$

$H(r)$ asymptotes to 1 as $r\to\infty$ so that the metric is asymptotically flat. In the 'near horizon', small $r$ limit the metric becomes

$$\mathrm{d}s^2 \;\simeq\; \frac{r^2}{L^2}\mathrm{d}x_4^2+L^2\frac{\mathrm{d}r^2}{r^2}+L^2\mathrm{d}\Omega_5^2\,. \tag{1.10}$$

The first two terms are that of an $\mathrm{AdS}_5$ metric while the last term is the metric of $S^5$. The curvature radius is $L$ for both metrics. When $g_sN\gg 1$ the curvature radius is large compared to the string length $\ell_s^2=\alpha'$ so the supergravity theory is a valid effective description but it breaks down when $g_sN\ll 1$. As in the first picture there are massless closed strings away from the brane. Near the horizon there are states of low energy because of the large redshift factor; these include the massless states but also the massive string states.

'Merging' the two viewpoints we arrive at the conjectured equivalence between $\mathcal{N}=4$ $SU(N)$ super-Yang Mills and type IIB string theory on $\mathrm{AdS}_5\times S^5$. The super-Yang-Mills theory has two parameters, the number of colors $N$ and the gauge coupling $g_{YM}^2$. On the string theory side these correspond to the amount of flux through $S^5$ and the string coupling $g_s$ respectively. Depending on the range of the parameters there are various limits of the correspondence which we summarize in table 1.3.

The symmetries on the two sides match as follows. The bosonic subgroup of the superconformal group, under which super-Yang Mills is invariant, is $SO(4,2)\times SU(4)$ where $SO(4,2)$ is the group of conformal transformations in four dimensions and $SU(4)\simeq SO(6)$ is the R-symmetry group. This corresponds to the isometries of $\mathrm{AdS}_5$ and $S^5$ respectively. In addition, the superconformal group in four dimensions possesses 32 supercharges and this is the number of supercharges conserved by $\mathrm{AdS}_5\times S^5$ in type IIB supergravity.

According to the holographic dictionary formulated in [5, 6] the gauge theory resides on the boundary at infinity of anti-deSitter space. The boundary values $\phi_0$ of fields $\phi$

propagating in the bulk of anti-deSitter space play the role of sources for dual operators in the gauge theory. This leads to the relation

$$\left\langle e^{\int d^d x\, \phi_0(x)\mathcal{O}(x)} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{string}}[\phi_0(x)] \,, \tag{1.11}$$

where the left-hand side is the generating functional of correlation function for the operator $\mathcal{O}$ and the right-hand side is the string theory partition function with boundary value $\phi_0$ for the $\phi$ field in AdS. In the supegravity approximation

$$Z_{\mathrm{string}}[\phi_0(x)] \simeq e^{-S_{\mathrm{sugra}}[\phi_0]} \,, \tag{1.12}$$

where $S_{\mathrm{sugra}}$ is the supergravity action. Equations (1.11) and (1.12) result in the free energy $\mathcal{F} = -\log |Z_{\mathrm{CFT}}|$ of the CFT being equal to the the action $S_{\mathrm{sugra}}$ evaluated on the dual supergravity background.

In general the AdS/CFT correspondence can be defined on backgrounds of string or M theory whose geometry possesses an $SO(d+2,2)$ symmetry; such backgrounds are (warped) products of anti-deSitter space and a compact manifold $Y$ and $p$-form potentials or fluxes which preserve the $SO(d+2,2)$ symmetry. Furthermore, the AdS/CFT correspondence is expected to hold for spacetimes which are only asymptotically anti-deSitter, such as anti-deSitter black holes which are dual to quantum field theories at finite temperature. It is thus of great interest to study the properties of anti-deSitter vacua of string and M-theory and deformations of anti-deSitter spacetime.

Extensively studied anti-deSitter backgrounds of supergravity theories are direct products of anti-deSitter spacetime with an Einstein manifold $Y$ (or an orbifold thereof). Also known as Freund-Rubin compactifications, this class of backgrounds constitutes a first deviation from the case where $Y$ is the round sphere, and arise as the near horizon limit of branes placed on the singularity of the cone $C(Y) \simeq \mathbb{R}_+ \times Y$ over $Y$. Requirement of supersymmetry translates into $C(Y)$ being a manifold of special holonomy [7] or equivalently $Y$ admitting Killing spinors i.e. spinors $\epsilon$ satisfying

$$\nabla_\mu \epsilon = \frac{1}{2}\gamma_\mu \epsilon \,. \tag{1.13}$$

In the case of $AdS_5 \times Y_5$ backgrounds $Y_5$ are Sasaki-Einstein manifolds, while in the case of $AdS_4 \times Y_7$ backgrounds, Einstein manifolds $Y_7$ admitting Killing spinors together with the associated cones $C(Y_7)$ are summarized in table 1.4.

Considerable research effort has been devoted to a systematic search and classification of generic supersymmetric anti-deSitter vacua and has produced important results which add to the outcome of the research on the field theory side of the correspondence. Comprehensive studies of general supersymmetric AdS geometries, in different dimensions, have appeared in [8, 9, 10, 11, 12]. The classification relied on the method of $G$-structures [13] which has proven a powerful mathematical tool in studying super-

| $C(Y_7)$ | $Y_7$ | supercharges |
|---|---|---|
| $\mathbb{R}^8$ | $S^7$ | $(8,8)$ |
| hyper-Kälher | three-Sasaki | $(3,0)$ |
| Calabi-Yau | Sasaki-Einstein | $(2,0)$ |
| $Spin(7)$ | weak $G_2$ holonomy | $(1,0)$ |

Table 1.4: Seven-manifolds admitting Killing spinors

symmetric backgrounds of supergravity theories.

**M2 branes and AdS$_4$/CFT$_3$**

Already in the renowned paper of Maldacena [4], it was suggested that M-theory on AdS$_4$ $\times$ $S^7$ i.e. the near-horizon limit of the membrane solution of eleven-dimensional supergravity, is dual to the field theory capturing the low-energy dynamics of M2 branes. Despite early efforts to formulate an AdS$_4$/CFT$_3$ correspondence [14] the latter remained underdeveloped due the elusive nature of the field theories living on the worldvolume of M2 branes.

This situation changed radically with the seminal work of [15, 16, 17] which identified the field theories describing the dynamics of multiple M2-branes and sparked great progress in understanding the AdS$_4$/CFT$_3$ correspondence in M-theory.

The theory proposed in [15, 16] as a worldvolume description of multiple M2-branes is based on a non-associative generalization of Lie algebra, a 3-algebra. The theory has $\mathcal{N} = 8$ supersymmetry with manifest $SO(8)$ R-symmetry. The theory introduced in [17], known as the ABJM theory, describes the low energy dynamics of M2-branes placed at the origin of the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. ABJM theory is a three-dimensional Chern-Simons-matter theory whose gauge group is $U(N) \times U(N)$, with Chern-Simons terms of opposite levels $k$ and $-k$ for each unitary group. The matter content comprises four chiral multiplets in the bifundamental representations $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ of the gauge group; it is summarised in the quiver diagram 1.2. ABJM theory possesses $\mathcal{N} = 6$ supersymmetry which is enhanced to $\mathcal{N} = 8$ for $k = 1$ or $k = 2$. In the latter case ABJM coincides with the theory constructed in [15, 16]. The gravity dual of ABJM theory is M-theory on AdS$_4$ $\times$ $S^7/\mathbb{Z}_k$ with $N$ units of flux threading AdS$_4$:

$$
\begin{aligned}
ds^2_{11} &= \tfrac{1}{4}L^2 ds^2_{\mathrm{AdS}_4} + L^2 ds^2_{S^7/\mathbb{Z}_k}\,, \qquad L^6 \;=\; 2^5\pi^2 k N \ell_p^6 \\
G &\propto N \mathrm{vol}_{\mathrm{AdS}_4}\,.
\end{aligned}
\tag{1.14}
$$

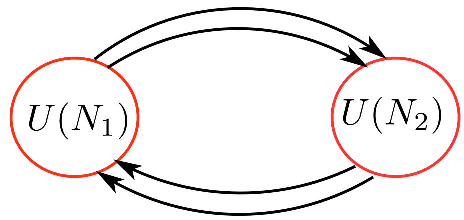


Figure 1.2: The quiver diagram of ABJM theory

Here $\ell_p$ is the Planck length.

Following this breakthrough many $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities were discovered including infinite families [18] - [26]. On the gravity side, the simplest setup is that of Freund-Rubin $\mathrm{AdS}_4 \times \mathrm{SE}_7$ backgrounds of M-theory, where $\mathrm{SE}_7$ is a Sasaki-Einstein manifold[2]. These are conjectured to be dual to the theory on a large number $N$ of M2-branes placed at a Calabi-Yau four-fold singularity. Rather generally, these field theories are believed to be strongly coupled Chern-Simons-matter theories at a conformal fixed point.

Thus far, almost all attention has been focused on $\mathrm{AdS}_4 \times \mathrm{SE}_7$ solutions. This is for the simple reason that very few $\mathrm{AdS}_4$ solutions outside this class are known. An exception is the Corrado-Pilch-Warner solution [27], which describes the infrared fixed point of a massive deformation of the maximally supersymmetric ABJM theory on $N$ M2-branes in flat spacetime. This solution is topologically $\mathrm{AdS}_4 \times S^7$, but the metric on $S^7$ is not round, and there is a non-trivial warp factor and internal four-form flux on the $S^7$. The group of isometries of $S^7$ is $SU(3) \times U(1) \times U(1)$ but the four form flux is charged under one of the $U(1)$'s and thus breaks this symmetry group to $SU(3) \times U(1)$. The dual gauge theory has been recently studied in [25, 28, 29], and in particular in [25] the free energy $\mathcal{F}$ of the $\mathcal{N} = 2$ superconformal fixed point was shown to match the free energy computed using the gravity dual solution.

The Corrado-Pilch-Warner solution has a simple generalization to massive deformations of $N$ M2-branes at a $\mathrm{CY}_3 \times \mathbb{C}$ four-fold singularity, where $\mathrm{CY}_3$ denotes an arbitrary Calabi-Yau three-fold cone singularity. Along these lines, $\mathrm{AdS}_4 \times S^7$ solutions with $SU(2) \times U(1)^2$ and $U(1)^3$ isometry were constructed in [30] and [31] respectively.

However, a systematic classification of the most general $\mathcal{N} = 2$ supersymmetric $\mathrm{AdS}_4$ solutions of M-theory with non-zero M2 brane charge, beyond the Freund-Rubin class, and a study of their properties in the context of $\mathrm{AdS}_4/\mathrm{CFT}_3$ was absent. It is one of the aims of the present thesis to provide one.

#### Rigid supersymmetry in curved space and holography

While gravity computations are relatively amenable, obtaining results directly in the three-dimensional strongly coupled field theories has been prohibitively difficult until

[2]Particular cases with $\mathcal{N} > 2$ include three-Sasakian manifolds and orbifolds of the round seven-sphere.

recently. For this reason, non-trivial quantitative tests of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence were not available. This situation has improved considerably with the results of [32, 33, 34] who showed that the partition function $Z$ of $\mathcal{N}=2$ supersymmetric field theories on the three-sphere can be reduced to a finite-dimensional matrix integral using localization techniques.

The basic idea of localization is the following. Given a supersymmetric theory with action $S(\phi)$ and a supercharge $Q$, one can deform the partition function as

$$Z(t) \;=\; \int D\phi \, \mathrm{e}^{-S(\phi)-t\delta_Q V} \;, \tag{1.15}$$

where $V$ is an operator invariant under $\delta_Q^2$. In fact $Z(t)$ is independent of $t$ since

$$\frac{\mathrm{d}Z}{\mathrm{d}t} \;=\; -\int D\phi \, \delta_Q V \mathrm{e}^{-S(\phi)-t\delta_Q V} \;=\; -\int D\phi \, \delta_Q \left( V \mathrm{e}^{-S(\phi)-t\delta_Q V} \right) \;=\; 0 \;. \tag{1.16}$$

In the final step in order to interpret the integrand as a total derivative one uses the fact that $\delta_Q$ is a symmetry of the path integral. The partition function $Z(t)$ can then be computed at $t=0$ (where one recovers the original partition function) but also for other values of $t$, like $t \to \infty$. In this regime, the partition function simplifies; if $V$ has a positive definite bosonic part $(\delta_Q V)_b$, the limit $t \to \infty$ localizes the integral to a submanifold of field space where

$$(\delta_Q V)_b \;=\; 0 \;. \tag{1.17}$$

It turns out that in many cases this submanifold is finite-dimensional. The same argument applies to supersymmetric observables such as supersymmetric Wilson loops.

More generally localization techniques can be applied to supersymmetric field theories on compact curved backgrounds other than the three-sphere, thus motivating the systematic study of rigid supersymmetry in curved space. In three dimensions the conditions for $\mathcal{N}=2$ supersymmetric field theories defined on a curved background have been studied in [35, 36] following the approach of [37]. The latter consists of coupling the field theory to an off-shell formulation of supegravity and taking the rigid limit by sending the Planck mass to infinity. The theory then resides on a curved space in the presence of background fields coming from off-shell supergravity. In particular there is a background gauge field which couples to the R-symmetry current. Supersymmetry is preserved if there exists a spinor parameter such that the variation of the gravitini under supersymmetry transformations vanish.

Prior to the systematic analysis of [35, 36] examples of rigid supersymmetric backgrounds other than the round three-sphere were constructed in [38] and [39]. In particular, a $U(1)\times U(1)$-symmetric background comprising a one-parameter squashed three-sphere was presented in [38] and two different $SU(2)\times U(1)$-symmetric backgrounds comprising a biaxially squashed three-sphere were presented in [38] and [39] respectively. These backgrounds include an abelian gauge field coupling to the R-symmetry current of the field theory. In all cases the partition function of an $\mathcal{N} = 2$ supersymmetric gauge theory defined on these backgrounds can be computed exactly using localization, and reduces to a matrix model involving the double sine function $s_\beta(z)$, where $\beta$ is related to the squashing parameter.

When a field theory defined on (conformally) flat space admits a gravity dual, it is natural to extend the holographic duality to cases where this field theory can be put on a non-trivial curved background. It is expected that field theories on curved manifolds correspond to deformations of anti-deSitter space i.e. asymptotically locally anti-deSitter spacetimes . In the present thesis we explore such holographic duals and compare their properties against those of the dual field theories.

*The rest of the thesis is based on [40] and [41].*

# 2 $\mathcal{N} = 2$ supersymmetric AdS$_4$ solutions of M-theory

## 2.1 Preview

In this chapter we systematically study the most general class of $\mathcal{N} = 2$ supersymmetric AdS$_4$ solutions of M-theory. These have an eleven-dimensional metric which is a warped product of AdS$_4$ with a compact Riemannian seven-manifold $Y_7$. In order that the $SO(3, 2)$ isometry group of AdS$_4$ is a symmetry group of the full solution, the four-form field strength necessarily has an "electric" component proportional to the volume form of AdS$_4$, and a "magnetic" component which is a pull-back from $Y_7$. We show, with the exception of the Sasaki-Einstein case, that the geometry on $Y_7$ admits a canonical local $SU(2)$-structure, and determine the necessary and sufficient conditions for a supersymmetric solution in terms of this structure. In particular, $Y_7$ is equipped with a canonical Killing vector field $\xi$, which is the geometric counterpart to the u(1) R-symmetry of the dual $\mathcal{N} = 2$ superconformal field theory.

Purely magnetic solutions correspond physically to wrapped M5-brane solutions, and we correspondingly recover the supersymmetry equations in [42] from our analysis. There is a single known solution in the literature, where $Y_7$ is an $S^4$ bundle over a three-manifold $\Sigma_3$ equipped with an Einstein metric of negative Ricci curvature. On the other hand, solutions with non-vanishing electric flux have a non-zero quantized M2-brane charge $N \in \mathbb{N}$, and include the Sasaki-Einstein manifold solutions as a special case where the magnetic flux vanishes. For the general class of solutions with non-vanishing M2-brane charge, we show that supersymmetry endows $Y_7$ with a canonical contact structure, for which the R-symmetry vector field $\xi$ is the unique Reeb vector field. A number of physical quantities can then be expressed purely in terms of contact volumes, including the gravitational free energy referred to above, and the scaling dimension of BPS operators $\mathcal{O}_{\Sigma_5}$ dual to probe M5-branes wrapped on supersymmetric five-submanifolds $\Sigma_5 \subset Y_7$. These formulae may be evaluated using topological and localization methods, allowing one to compute the free energy and scaling dimensions of certain BPS operators without knowing the detailed form of the supergravity solution.

In our analysis we recover the Corrado-Pilch-Warner solution as a solution to our system of $SU(2)$-structure equations. We also show that this solution is in a subclass of solutions which possess an additional Killing vector field. For this subclass the supersymmetry conditions are equivalent to specifying a (local) Kähler-Einstein four-metric, together with a solution to a particular second order non-linear ODE. We show that this ODE admits a solution with the correct boundary conditions to give a gravity dual to the infrared fixed point of cubic deformations of $N$ M2-branes at a $\mathrm{CY}_3\times\mathbb{C}$ four-fold singularity. In particular, when $\mathrm{CY}_3=\mathbb{C}^3$ equipped with its flat metric, this leads to a new, smooth $\mathcal{N}=2$ supersymmetric $\mathrm{AdS}_4\times S^7$ solution of M-theory.

## 2.2 The conditions for supersymmetry

In this section we analyse the general conditions for $\mathcal{N}=2$ supersymmetry for a warped $\mathrm{AdS}_4\times Y_7$ background of eleven-dimensional supergravity.

### 2.2.1 Ansatz and spinor equations

The bosonic fields of eleven-dimensional supergravity consist of a metric $g_{11}$ and a three-form potential $C$ with four-form field strength $G=\mathrm{d}C$. The signature of the metric is $(-,+,+,\ldots,+)$ and the action is

$$S \;=\; \frac{1}{(2\pi)^8\ell_p^9}\int\left(R *_{11} 1-\frac{1}{2}G\wedge *_{11}G-\frac{1}{6}C\wedge G\wedge G\right)\ , \tag{2.1}$$

with $\ell_p$ the eleven-dimensional Planck length. The resulting equations of motion are

$$\begin{aligned} R_{MN}-\frac{1}{12}\left[G_{MPQR}G_N{}^{PQR}-\frac{1}{12}(g_{11})_{MN}G^2\right] &= 0\ ,\\ \mathrm{d}*_{11}G+\frac{1}{2}G\wedge G &= 0\ , \end{aligned} \tag{2.2}$$

where $M,N=0,\ldots,10$ denote spacetime indices.

We consider AdS$_4$ solutions of M-theory of the warped product form

$$\begin{aligned} g_{11} &= \mathrm{e}^{2\Delta}\left(g_{\mathrm{AdS}_4}+g_7\right)\ ,\\ G &= m\mathrm{vol}_4+F\ . \end{aligned} \tag{2.3}$$

Here vol$_4$ denotes the Riemannian volume form on AdS$_4$, and without loss of generality we take $\mathrm{Ric}_{\mathrm{AdS}_4}=-12g_{\mathrm{AdS}_4}$.[1] In order to preserve the $SO(3,2)$ invariance of AdS$_4$ we take the warp factor $\Delta$ to be a function on the compact seven-manifold $Y_7$, and $F$ to be the pull-back of a four-form on $Y_7$. The Bianchi identity $\mathrm{d}G=0$ then requires that $m$ is constant. The case in which $m\neq 0$ will turn out to be quite distinct from that with $m=0$.

[1] The factor here is chosen to coincide with standard conventions in the case that $Y_7$ is a Sasaki-Einstein seven-manifold. For example, the AdS$_4$ metric in global coordinates then reads $g_{\mathrm{AdS}_4}=\frac{1}{4}(-\cosh^2\varrho\,\mathrm{d}t^2+\mathrm{d}\varrho^2+\sinh^2\varrho\,\mathrm{d}\Omega_2^2)$, where $\mathrm{d}\Omega_2^2$ denotes the unit round metric on $S^2$.

In an orthonormal frame, the Clifford algebra Cliff(10, 1) is generated by gamma matrices $\Gamma_A$ satisfying $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, where the frame indices $A, B = 0, \ldots, 10$, and $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$, and we choose a representation with $\Gamma_0 \cdots \Gamma_{10} = 1$. The Killing spinor equation is

$$\nabla_M \epsilon + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\delta_M{}^N \Gamma^{PQR}\right) G_{NPQR}\, \epsilon \;=\; 0 \;, \tag{2.4}$$

where $\epsilon$ is a Majorana spinor. We may decompose Cliff(10, 1) $\cong$ Cliff(3, 1) $\otimes$ Cliff(7, 0) via

$$\Gamma_\alpha \;=\; \hat{\gamma}_\alpha \otimes 1 \;, \qquad \Gamma_{a+3} \;=\; \hat{\gamma}_5 \otimes \gamma_a \;, \tag{2.5}$$

where $\alpha, \beta = 0, 1, 2, 3$ and $a, b = 1, \ldots, 7$ are orthonormal frame indices for AdS$_4$ and $Y_7$ respectively, $\{\hat{\gamma}_\alpha, \hat{\gamma}_\beta\} = 2\eta_{\alpha\beta}$, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, and we have defined $\hat{\gamma}_5 = \mathrm{i}\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$. Notice that our eleven-dimensional conventions imply that $\gamma_1 \cdots \gamma_7 = \mathrm{i}1$.

The spinor ansatz preserving $\mathcal{N}=1$ supersymmetry in AdS$_4$ is

$$\epsilon \;=\; \psi^+ \otimes \mathrm{e}^{\Delta/2}\chi + (\psi^+)^c \otimes \mathrm{e}^{\Delta/2}\chi^c \;, \tag{2.6}$$

where $\psi^+$ is a positive chirality Killing spinor on AdS$_4$, so $\hat{\gamma}_5\psi^+ = \psi^+$, satisfying

$$\nabla_\mu \psi^+ \;=\; \hat{\gamma}_\mu (\psi^+)^c \;. \tag{2.7}$$

The superscript $c$ in (2.6) denotes charge conjugation in the relevant dimension, and the factor of $\mathrm{e}^{\Delta/2}$ is included for later convenience. Substituting (2.6) into the Killing spinor equation (2.4) leads to the following algebraic and differential equations for the spinor field $\chi$ on $Y_7$.

$$\begin{aligned} \frac{1}{2}\gamma^n \partial_n \Delta \chi - \frac{\mathrm{i}m}{6}\mathrm{e}^{-3\Delta}\chi + \frac{1}{288}\mathrm{e}^{-3\Delta} F_{npqr}\gamma^{npqr}\chi + \chi^c &\;=\; 0 \;, \\ \nabla_m \chi + \frac{\mathrm{i}m}{4}\mathrm{e}^{-3\Delta}\gamma_m \chi - \frac{1}{24}\mathrm{e}^{-3\Delta} F_{mpqr}\gamma^{pqr}\chi - \gamma_m \chi^c &\;=\; 0 \;. \end{aligned} \tag{2.8}$$

For a supergravity solution one must also solve the equations of motion (2.2) resulting from (2.1), as well as the Bianchi identity $\mathrm{d}G = 0$.

Motivated by the discussion in the introduction, we will focus on $\mathcal{N}=2$ supersymmetric AdS$_4$ solutions for which there are two independent solutions $\chi_1$, $\chi_2$ to (2.8). The general $\mathcal{N}=2$ Killing spinor ansatz may be written as

$$\epsilon \;=\; \sum_{i=1,2}\left(\psi_i^+ \otimes \mathrm{e}^{\Delta/2}\chi_i + (\psi_i^+)^c \otimes \mathrm{e}^{\Delta/2}\chi_i^c\right) \;. \tag{2.9}$$

In general the two Killing spinors $\psi_i^+$ on AdS$_4$ satisfy an equation of the form

$$\nabla_\mu \psi_i^+ \quad = \quad \sum_{j=1}^{2} W_{ij} \hat{\gamma}_\mu (\psi_j^+)^c \ . \tag{2.10}$$

Multiplying by $\bar{\psi}_k^+ \hat{\gamma}^\mu$ on the left it is not difficult to show that $W_{ij}$ is necessarily a constant matrix. Using the integrability conditions of (2.10),

$$\sum_j W_{ij} W_{jk}^* \quad = \quad \delta_{ik} \ , \tag{2.11}$$

one can verify that, without loss of generality, by a change of basis we may take $W_{ij} = \delta_{ij}$ to be the identity matrix. Thus $\psi_1^+$ and $\psi_2^+$ may both be taken to satisfy (2.7).

In this case with $\mathcal{N} = 2$ supersymmetry there is a $u(1)$ R-symmetry which rotates the spinors as a doublet. It is then convenient to introduce

$$\chi_\pm \quad \equiv \quad \frac{1}{\sqrt{2}} (\chi_1 \pm \mathrm{i}\chi_2) \ , \tag{2.12}$$

which will turn out to have charges $\pm 2$ under the Abelian R-symmetry. In terms of the new basis (2.12), the spinor equations (2.8) read

$$\begin{aligned} \frac{1}{2}\gamma^n \partial_n \Delta \chi_\pm - \frac{\mathrm{i}m}{6} \mathrm{e}^{-3\Delta} \chi_\pm + \frac{1}{288} \mathrm{e}^{-3\Delta} F_{npqr} \gamma^{npqr} \chi_\pm + \chi_\mp^c \quad &= \quad 0 \ , \\ \nabla_m \chi_\pm + \frac{\mathrm{i}m}{4} \mathrm{e}^{-3\Delta} \gamma_m \chi_\pm - \frac{1}{24} \mathrm{e}^{-3\Delta} F_{mpqr} \gamma^{pqr} \chi_\pm - \gamma_m \chi_\mp^c \quad &= \quad 0 \ . \end{aligned} \tag{2.13}$$

### 2.2.2 Preliminary analysis

The condition of $\mathcal{N} = 2$ supersymmetry means that the spinors $\chi_1$, $\chi_2$ in (2.9) are linearly independent. Notice that we are free to make $GL(2, \mathbb{R})$ transformations of the pair $(\chi_1, \chi_2)$, since this leaves the spinor equations (2.13) invariant. We shall make use of this freedom below.

The scalar bilinears are $\bar{\chi}_i \chi_j$ and $\bar{\chi}_i^c \chi_j$, which may equivalently be rewritten in the $\chi_\pm$ basis (2.12). The differential equation in (2.8) immediately gives $\nabla(\bar{\chi}_1 \chi_1) = \nabla(\bar{\chi}_2 \chi_2) = 0$, so that using $\mathbb{R}^* \times \mathbb{R}^* \subset GL(2, \mathbb{R})$ we may without loss of generality set $\bar{\chi}_1 \chi_1 = \bar{\chi}_2 \chi_2 = 1$. Setting $\mathcal{C} = 1$ in (A.3), the algebraic equation in (2.8) thus leads to

$$2\mathrm{Im} \left[ \bar{\chi}_i^c \chi_j \right] \quad = \quad -\frac{m}{3} \mathrm{e}^{-3\Delta} \bar{\chi}_i \chi_j \ , \tag{2.14}$$

where $i, j \in \{1, 2\}$. We immediately conclude that for $m \neq 0$ we have

$$\mathrm{Im} [\bar{\chi}_1 \chi_2] \quad = \quad 0 \ . \tag{2.15}$$

When $m = 0$ this statement is not necessarily true. The case with $m = 0$ and $\mathrm{Im}[\bar{\chi}_1 \chi_2]$

not identically zero is discussed separately in subsection 2.5.2, where we show that there are no regular solutions in this class. We may therefore take (2.15) to hold in all cases.

It is straightforward to analyse the remaining scalar bilinear equations. In particular, $\mathrm{Re}\,[\bar{\chi}_1\chi_2]$ is constant, and using the remaining $GL(2,\mathbb{R})$ freedom one can without loss of generality set $\mathrm{Re}\,[\bar{\chi}_1\chi_2]=0$.[2] In the $\chi_\pm$ basis (2.12) we may then summarize the results of this analysis as

$$\begin{aligned} \bar{\chi}_+\chi_+ &= 1 = \bar{\chi}_-\chi_- \,, \qquad & \bar{\chi}_+\chi_- &= 0\,, \\ \bar{\chi}^c_+\chi_+ &\equiv S = (\bar{\chi}^c_-\chi_-)^*\,, \qquad & \bar{\chi}^c_+\chi_- &= -\mathrm{i}\frac{m}{6}\mathrm{e}^{-3\Delta}\,. \end{aligned} \tag{2.16}$$

where $S$ is a complex function on $Y_7$. We also define the one-form bilinears

$$\begin{aligned} K &\equiv \mathrm{i}\bar{\chi}^c_+\gamma_{(1)}\chi_-\,, \qquad L \equiv \bar{\chi}_-\gamma_{(1)}\chi_+\,, \\ \bar{\chi}_+\gamma_{(1)}\chi_+ &\equiv -P = -\bar{\chi}_-\gamma_{(1)}\chi_-\,. \end{aligned} \tag{2.17}$$

Here we have denoted $\gamma_{(n)}\equiv\frac{1}{n!}\gamma_{m_1\cdots m_n}\mathrm{d}y^{m_1}\wedge\cdots\wedge\mathrm{d}y^{m_n}$. *A priori* notice that $K$ and $L$ are complex, while $P$ is real.

### 2.2.3 The R-symmetry Killing vector

The spinor equations (2.13) imply that

$$2\mathrm{Im}\,K = \mathrm{d}\mathrm{Im}\,[\bar{\chi}_1\chi_2] = 0\,, \tag{2.18}$$

where we have used (2.15). Thus in fact $K$ is real, and it is then straightforward to show that $K$ is a Killing one-form for the metric $g_7$ on $Y_7$, and hence that the dual vector field $\xi\equiv g_7^{-1}(K,\cdot)$ is a Killing vector field. More precisely, one computes

$$\nabla_{(m}K_{n)} = -2\mathrm{i}\,\mathrm{Im}\,[\bar{\chi}_1\chi_2]\,g_{7\,mn} = 0\,. \tag{2.19}$$

Using the Fierz identity (A.6) one computes the square norm

$$\|\xi\|^2 \equiv g_7(\xi,\xi) = |S|^2+\frac{m^2}{36}\mathrm{e}^{-6\Delta}\,. \tag{2.20}$$

In particular when $m\neq0$ we see that $\xi$ is nowhere zero, and thus defines a one-dimensional foliation of $Y_7$. In the case that $m=0$ this latter conclusion is no longer true in general, as we will show in section 2.2.7 via a counterexample.

The algebraic equation in (2.13) leads immediately to $\mathcal{L}_\xi\Delta=0$, and using both

[2] In the special case that $\mathrm{Re}\,[\bar{\chi}_1\chi_2]=1$ one can show that $\chi_1=\chi_2$, which in turn leads to only $\mathcal{N}=1$ supersymmetry.

equations in (2.13) one can show that

$$\mathrm{d}(\mathrm{e}^{3\Delta}\,\bar{\chi}_+^c\gamma_{(2)}\chi_-) \;\;=\;\; -\mathrm{i}\xi\lrcorner F\ . \tag{2.21}$$

It follows that

$$\mathcal{L}_\xi F \;\;=\;\; \mathrm{d}(\xi\lrcorner F) + \xi\lrcorner \mathrm{d}F \;=\; 0\ , \tag{2.22}$$

provided the Bianchi identity $\mathrm{d}F = 0$ holds.[3] Thus $\xi$ preserves all of the bosonic fields.

One can also show that

$$\mathcal{L}_\xi\chi_\pm \;\;=\;\; \pm 2\mathrm{i}\,\chi_\pm\ , \tag{2.23}$$

so that $\chi_\pm$ have charges $\pm 2$ under $\xi$. Perhaps the easiest way to prove this is to use the remaining non-trivial scalar bilinear equation

$$\mathrm{e}^{-3\Delta}\mathrm{d}(\mathrm{e}^{3\Delta}S) \;\;=\;\; 4L\ , \tag{2.24}$$

to show that

$$\mathcal{L}_\xi S \;\;=\;\; 4\mathrm{i}S\ . \tag{2.25}$$

Since $\xi$ preserves all of the bosonic fields, we may take the Lie derivative of the spinor equations (2.13) to conclude that $\mathcal{L}_\xi\chi_\pm$ satisfy the *same* equations, and hence $\mathcal{L}_\xi\chi_\pm$ are linear combinations of $\chi_\pm$. The Lie derivatives of the scalar bilinears, in particular (2.25), then fix (2.23).[4] We thus identify $\xi$ as the canonical vector field dual to the R-symmetry of the $\mathcal{N} = 2$ SCFT.

### 2.2.4 Equations of motion

Given our ansatz, the equation of motion and Bianchi identity for $G$ reduce to

$$\mathrm{d}\left(\mathrm{e}^{3\Delta}\star F\right) \;\;=\;\; -mF\ , \qquad \mathrm{d}F \;=\; 0\ , \tag{2.26}$$

where $\star$ denotes the Hodge star operator on $Y_7$. We begin by showing that supersymmetry implies the equation of motion, and that for $m \neq 0$ it also implies the Bianchi identity.

The imaginary part of the bilinear equation for the three-form $\bar{\chi}_+^c\gamma_{(3)}\chi_-$ leads immediately to

$$mF \;\;=\;\; 6\,\mathrm{d}\left(\mathrm{e}^{6\Delta}\mathrm{Im}\left[\bar{\chi}_+^c\gamma_{(3)}\chi_-\right]\right)\ . \tag{2.27}$$

[3] In fact this is implied by supersymmetry when $m \neq 0$, as we will show shortly in section 2.2.4.

[4] More precisely, this argument is valid provided $S$ is not identically zero. However, when $S = 0$ we necessarily reduce to the Sasaki-Einstein case, as shown in subsection 2.5.1. In that case (2.23) also holds.

Thus for $m \neq 0$ we deduce that $F$ is closed. On the other hand, the bilinear equation for the two-form $\bar{\chi}_+\gamma_{(2)}\chi_+$

$$e^{3\Delta} \star F \quad = \quad \mathrm{d}\left(\mathrm{i}\, e^{6\Delta}\bar{\chi}_+\gamma_{(2)}\chi_+\right) - 6e^{6\Delta}\mathrm{Im}\left[\bar{\chi}_+^c\gamma_{(3)}\chi_-\right] \ , \tag{2.28}$$

gives, via taking the exterior derivative,

$$\mathrm{d}\left(e^{3\Delta}\star F\right) \quad = \quad -6\mathrm{d}\left(e^{6\Delta}\mathrm{Im}\left[\bar{\chi}_+^c\gamma_{(3)}\chi_-\right]\right) \quad = \quad -mF \ , \tag{2.29}$$

where in the second equality we have combined with equation (2.27). We thus see that supersymmetry implies the equation of motion in (2.26).

Finally, using the integrability results of [43] one can now show that the Einstein equation is automatically implied as an integrability condition for the supersymmetry conditions, once the $G$-field equation and Bianchi identity are imposed. In particular, note that the eleven-dimensional one-form bilinear $k \equiv \bar{\epsilon}\Gamma_{(1)}\epsilon$ is dual to a timelike Killing vector field, as discussed in [44] and later in subsection 2.3.4. We thus conclude

> *For the class of* $\mathcal{N}=2$ *supersymmetric* AdS$_4$ *solutions of the form* (2.3)*, supersymmetry and the Bianchi identity* $\mathrm{d}F=0$ *imply the equations of motion for G and the Einstein equations. Moreover, when* $m \neq 0$ *the Bianchi identity* $\mathrm{d}F=0$ *is also implied by supersymmetry.*

Note that similar results were obtained in both [9] and [12]. In fact we will see in subsection 2.2.7 that the $m=0$ supersymmetry conditions also imply the Bianchi identity, although the arguments we have presented so far do not allow us to conclude this yet.

### 2.2.5 Introducing a canonical frame

Provided the three real one-forms $K, \mathrm{Re}\, S^*L, \mathrm{Im}\, S^*L$ defined in (2.17) are linearly independent, we may use them to in turn define a canonical orthonormal three-frame $\{e^5, e^6, e^7\}$[5]. More precisely, if these three one-forms are linearly independent at a point in $Y_7$, the stabilizer group $\mathcal{G} \subset \mathrm{Spin}(7)$ of the pair of spinors $\chi_\pm$ at that point is $\mathcal{G} \cong SU(2)$, giving a natural identification of the tangent space with $\mathbb{C}^2 \oplus \mathbb{R}e^5 \oplus \mathbb{R}e^6 \oplus \mathbb{R}e^7$. Here the $SU(2)$ structure group acts on $\mathbb{C}^2$ in the vector representation. If this is true in an open set, it will turn out that we may go further and also introduce three canonical coordinates associated to the three-frame $\{e^5, e^6, e^7\}$.[6]

We study the case that $K, \mathrm{Re}\, S^*L, \mathrm{Im}\, S^*L$ are linearly *dependent* in subsection 2.5.1. In particular, for $m \neq 0$ we conclude that at least one of $S=0$ or $\xi=1$ holds at such

[5] We use $S^*L$ here, as opposed to $L$, since $S^*L$ is invariant under the R-symmetry generated by $\xi$. In particular, from the definitions in (2.17), and using (2.23), (2.25), we have that $\mathcal{L}_\xi K = \mathcal{L}_\xi(S^*L) = 0$.

[6] Just from group theory it must be the case that the one-form $P$ in (2.17) is a linear combination of $K$ and $S^*L$, and indeed one finds that $\frac{m}{6}e^{-3\Delta}P = K + \mathrm{Im}\, SL^*$.

a point. If this is the case over the whole of $Y_7$ (or, using analyticity and connectedness, if this is the case on any open subset of $Y_7$) then we show that $Y_7$ is necessarily Sasaki-Einstein with $F=0$. Of course, in general the three one-forms can become linearly dependent over certain submanifolds of $Y_7$, and here our orthonormal frame and coordinates will break down.[7] By analogy with the corresponding situation for $\mathrm{AdS}_5$ solutions of type IIB string theory studied in [45], one expects this locus to be the same as the subspace where a pointlike M2-brane is BPS, and thus correspond to the Abelian moduli space of the dual CFT, although we will not pursue this comment further here.

Returning to the generic case in which $K, \mathrm{Re}\, S^*L, \mathrm{Im}\, S^*L$ are linearly independent in some region, we may begin by introducing a coordinate $\psi$ along the orbits of the Reeb vector field $\xi$, so that

$$\xi \quad \equiv \quad 4\frac{\partial}{\partial\psi} \ . \tag{2.30}$$

The equation (2.25) then implies that we may write

$$S \quad = \quad \mathrm{e}^{-3\Delta}\rho\, \mathrm{e}^{\mathrm{i}(\psi-\tau)} \ . \tag{2.31}$$

This defines the real functions $\rho$ and $\tau$, which will serve as two additional coordinates on $Y_7$. The factor of $\mathrm{e}^{-3\Delta}$ has been included partly for convenience, and partly to agree with conventions defined in [42] that we will recover from the $m=0$ limit in subsection 2.2.7. Using (2.24) together with the Fierz identity (A.6), one can then check that

$$\begin{aligned}
e^5 \quad &\equiv \quad \frac{1}{|S|\sqrt{1-\|\xi\|^2}}\mathrm{Re}\, S^*L \quad = \quad \frac{\mathrm{e}^{-3\Delta}}{4\sqrt{1-\|\xi\|^2}}\mathrm{d}\rho \ , \\
e^6 \quad &\equiv \quad \frac{6\mathrm{e}^{3\Delta}|S|}{m\,\|\xi\|\sqrt{1-\|\xi\|^2}}\left(K-\frac{\|\xi\|^2}{|S|^2}\mathrm{Im}\, S^*L\right) \quad = \quad \frac{3\mathrm{e}^{3\Delta}|S|\,\|\xi\|}{2m\sqrt{1-\|\xi\|^2}}(\mathrm{d}\tau+\mathscr{A}) \ , \\
e^7 \quad &\equiv \quad \frac{1}{\|\xi\|}K \quad = \quad \frac{1}{4}\|\xi\|\,(\mathrm{d}\psi+\mathscr{A}) \ ,
\end{aligned} \tag{2.32}$$

are orthonormal. Here $\mathscr{A}$ is a local one-form that is basic for the foliation defined by the Reeb vector field $\xi$, i.e. $\mathcal{L}_\xi\mathscr{A}=0$, $\xi\lrcorner\mathscr{A}=0$. Note here that

$$\|\xi\|^2 \quad \equiv \quad g_{Y_7}(\xi,\xi) \quad = \quad \frac{\mathrm{e}^{-6\Delta}}{36}(m^2+36\rho^2) \ , \tag{2.33}$$

is the square length of the Reeb vector field. The metric on $Y_7$ may then be written as

$$g_7 \quad = \quad g_{SU(2)}+(e^5)^2+(e^6)^2+(e^7)^2 \ . \tag{2.34}$$

[7] This is sometimes referred to as a dynamical $SU(2)$ structure.

We may now in turn introduce an orthonormal frame $\{e_a\}_{a=1}^4$ for $g_{SU(2)}$, and define the $SU(2)$-invariant two-forms

$$\begin{aligned} J &\equiv J_3 \equiv e^1\wedge e^2 + e^3\wedge e^4 \ , \\ \Omega &\equiv J_1 + \mathrm{i}J_2 \equiv (e^1+\mathrm{i}e^2)\wedge(e^3+\mathrm{i}e^4) \ . \end{aligned} \tag{2.35}$$

Of course, such a choice is not unique – we are free to make $SU(2)_R$ rotations, under which $J_I$, $I=1,2,3$, transform as a triplet, where the structure group is $\mathcal{G}\cong SU(2)=SU(2)_L$, and Spin(4) $\cong SU(2)_L\times SU(2)_R$ is the spin group associated to $g_{SU(2)}$.

### 2.2.6 Necessary and sufficient conditions

Any spinor bilinear may be written in terms of $\{e^i\}_{i=5}^7$, $J_I$, having chosen a convenient basis[8] for the $J_I$. Having solved for the one-forms in (3.14), the remaining differential conditions arising from $k$-form bilinears, for all $k\leq 3$, then be shown to reduce (after some lengthy computations) to the following system of three equations

$$\boxed{\begin{aligned} \mathrm{e}^{-3\Delta}\mathrm{d}\left[\|\xi\|^{-1}\left(\tfrac{m}{6}e^7+\mathrm{e}^{3\Delta}|S|\sqrt{1-\|\xi\|^2}e^6\right)\right] &= 2J_3 - 2\|\xi\|\, e^5\wedge e^6 \ , \\ \mathrm{d}(\|\xi\|^2\,\mathrm{e}^{9\Delta}J_2\wedge e^5) - \mathrm{e}^{3\Delta}|S|\mathrm{d}(\|\xi\|\,\mathrm{e}^{6\Delta}|S|^{-1}J_1\wedge e^6) &= 0 \ , \\ \mathrm{d}(\mathrm{e}^{6\Delta}J_1\wedge e^5) + \mathrm{e}^{3\Delta}|S|\mathrm{d}(\|\xi\|\,\mathrm{e}^{3\Delta}|S|^{-1}J_2\wedge e^6) &= 0 \ , \end{aligned}} \tag{2.36}$$

where in addition the flux is determined by the equation

$$\mathrm{d}(\mathrm{e}^{6\Delta}\sqrt{1-\|\xi\|^2}J_2) = -\mathrm{e}^{3\Delta}\star F - 6\mathrm{e}^{6\Delta}\mathrm{Im}\left[\bar{\chi}^c_+\gamma_{(3)}\chi_-\right] \ . \tag{2.37}$$

Notice this is the same equation (2.28) we already used in proving that the equation of motion for $G$ follows from supersymmetry. The bilinear on the right hand side is given in terms of our frame by

$$\mathrm{Im}\left[\bar{\chi}^c_+\gamma_{(3)}\chi_-\right] = |S|J_2\wedge e^5 - \frac{1}{\|\xi\|}J_1\wedge(\tfrac{m}{6}\mathrm{e}^{-3\Delta}\sqrt{1-\|\xi\|^2}e^7+|S|e^6) \ . \tag{2.38}$$

One can invert the expression for the flux using these equations to obtain

$$\boxed{F = \frac{1}{\|\xi\|}e^7\wedge\mathrm{d}\left(\mathrm{e}^{3\Delta}\sqrt{1-\|\xi\|^2}J_1\right) - m\frac{\sqrt{1-\|\xi\|^2}}{\|\xi\|}J_1\wedge e^5\wedge e^6 \ .} \tag{2.39}$$

Notice that although we have written these equations in terms of the two real functions $|S|$ and $\|\xi\|$, in fact they obey (2.33). Regarding $\rho$ as a coordinate, there is then really

[8] Notice that using the definition of the two-form bilinears in B, and the fact that $\mathcal{L}_\xi e^i=0$, we see that also the $J_I$ are invariant under $\xi$, namely $\mathcal{L}_\xi J_I=0$.

only one independent function in these equations, which may be taken to be the warp factor $\Delta$. We also note that the connection one-form $\mathscr{A}$, defined via the orthonormal frame (3.14), has curvature determined by the first equation in (2.36), giving

$$\mathrm{d}\mathscr{A} \quad = \quad \frac{4m\mathrm{e}^{-3\Delta}}{3\left\|\xi\right\|^2}\left[J_3 + \left(3\left\|\xi\right\| - \frac{4}{\left\|\xi\right\|}\right)e^5\wedge e^6\right] . \tag{2.40}$$

**Proof of sufficiency**

It is important to stress that the set of equations (2.36), where the three-frame $\{e^i\}_{i=5}^7$ is given by (2.32), are both necessary and sufficient for a supersymmetric solution. In order to see this, we recall that our $SU(2)$ structure can be thought of in terms of the two $SU(3)$ structures defined by the spinors $\chi_+$, $\chi_-$ (or equivalently $\chi_1$, $\chi_2$). Each of these determines a real vector $\mathcal{K}_\pm \equiv \bar{\chi}_\pm\gamma_{(1)}\chi_\pm$, real two-form $\mathcal{J}_\pm \equiv -\mathrm{i}\bar{\chi}_\pm\gamma_{(2)}\chi_\pm$, and complex three-form $\Omega_\pm \equiv \bar{\chi}^c_\pm\gamma_{(3)}\chi_\pm$, where recall that also $\bar{\chi}_+\chi_+ = \bar{\chi}_-\chi_- = 1$. In fact $\mathcal{K}_+ = -\mathcal{K}_- = -P$, so that the vectors determined by each $SU(3)$ structure are equal and opposite, and $(\mathcal{J}_\pm, \Omega_\pm)$ determine two $SU(3)$ structures on the transverse six-space $P^\perp$.

Let us now turn to the Killing spinor equations in (2.13). We have two copies of these equations, one for each $SU(3)$ structure determined by the spinors $\chi_\pm$. We shall refer to the first equation in (2.13) as the algebraic Killing spinor equation (it contains no derivative acting on the spinor itself). Using this notice that we may eliminate the $\chi^c_\mp$ term in the second equation, in order to get an equation linear in $\chi_\pm$; we shall refer the resulting equation as the differential Killing spinor equation. For each choice of $\pm$, the latter may be phrased in terms of a generalized connection $\nabla^{(T)}_\pm$, where $\nabla$ is the Levi-Civita connection. The intrinsic torsion is then defined as $\tau_\pm \equiv \nabla^{(T)}_\pm - \nabla$ for each $SU(3)$ structure, and may be decomposed into irreducible $SU(3)$-modules as a section of $\Lambda^1\otimes\Lambda^2$. Since $\Lambda^2 \cong \mathrm{so}(7) = \mathrm{su}(3)\oplus\mathrm{su}(3)^\perp$, the intrinsic torsion may be identified as a section of $\Lambda^1\otimes\mathrm{su}(3)^\perp$. It is then a fact that the exterior derivatives of $\mathcal{K}_\pm$, $\mathcal{J}_\pm$, $\Omega_\pm$ determine completely the intrinsic torsion $\tau_\pm$ - the identifications of the irreducible modules are given explicitly in section 2.3 of [46]. Our equations (2.36) certainly imply the exterior derivatives of both $SU(3)$ structures, since they imply the exterior derivatives of all $k$-form bilinears, for $k \leq 3$. It follows that from our supersymmetry equations we could (in principle) construct both $\tau_\pm$, and hence write down connections $\nabla^{(T)}_\pm = \nabla + \tau_\pm$ which preserve each spinor, so $\nabla^{(T)}_\pm\chi_\pm = 0$. In other words, our conditions then imply the differential Killing spinor equations for each of the $\mathcal{N} = 2$ supersymmetries.

For the algebraic Killing spinor equation, note first that $\{\chi, \gamma_m\chi \mid m = 1,\ldots,7\}$ forms a basis for the spinor space for each $\chi = \chi_\pm$. Thus in order for the algebraic equation to hold, it is sufficient that the bilinear equations resulting from the contraction of the algebraic Killing spinor equation with $\bar{\chi}$ and $\bar{\chi}\gamma_m$ hold, where $\chi$ is either of $\chi_\pm$. However, this is precisely how the identities in appendix A were derived. We thus find that the

$\chi_+$ algebraic Killing spinor equation in (2.13) is implied by the two zero-form equations

$$\begin{aligned} -\frac{m}{3}\mathrm{e}^{-3\Delta} + 2\mathrm{Im}\,\bar{\chi}_+\chi_-^c &= 0 \,, \\ \mathrm{d}\Delta \lrcorner \mathcal{K}_+ + \frac{1}{6}\mathrm{e}^{-3\Delta}\bar{\chi}_+\gamma_{(4)}\chi_+ \lrcorner F &= 0 \,, \end{aligned} \tag{2.41}$$

and the one-form equations

$$\begin{aligned} \mathrm{d}\Delta + \frac{1}{6}\mathrm{e}^{-3\Delta}\mathcal{J}_+ \lrcorner \star F &= 0 \,, \\ \frac{m}{3}\mathrm{e}^{-3\Delta}P - 2K + \mathcal{J}_+(\mathrm{d}\Delta) - \frac{1}{6}\mathrm{e}^{-3\Delta}(\mathrm{i}\bar{\chi}_+\gamma_{(3)}\chi_+)\lrcorner F &= 0 \,, \end{aligned} \tag{2.42}$$

with similar equations for $\chi_-$. Notice that the first equation in (2.41) is simply the scalar bilinear in (2.16). Using these expressions, one can show that (2.36) imply the remaining scalar equation in (2.41) and both of the equations in (2.42), thus proving that our differential system (2.36) also implies the algebraic Killing spinor equations. The computation is somewhat tedious, and is best done by splitting the equations (2.36) into components under the $1 + 1 + 1 + 4$ decomposition implied by the three-frame (2.32). This decomposition is performed explicitly in subsection 2.2.8. In the second equation in (2.41) we note that each term is in fact separately zero. We also note that the first equation in (2.42) may be rewritten as

$$\mathcal{J}_+ \lrcorner \mathrm{d}(\mathrm{e}^{6\Delta}\mathcal{J}_+) \;=\; \mathrm{d}\left(\mathrm{e}^{6\Delta}(1 - \tfrac{3}{2}|S|^2)\right) \,. \tag{2.43}$$

The left hand side is essentially the *Lee form* associated to the $SU(3)$-structure defined by $\chi_+$[9].

To conclude, we have shown that (2.36) are necessary and sufficient to satisfy the original Killing spinor equations (2.13).

### 2.2.7 M5-brane solutions: $m = 0$

It is straightforward to take the $m = 0$ limit of the frame (2.32), differential conditions (2.36), and flux $F$ given by (2.39). Denoting $\hat{w} = \mathrm{e}^{\Delta}e^6$, $\hat{\rho} = \mathrm{e}^{\Delta}e^5$, $\hat{J}_I = \mathrm{e}^{2\Delta}J_I$ and $\lambda = \mathrm{e}^{-2\Delta}$ we obtain the metric

$$\lambda^{-1}g_7 \;=\; \widehat{g_{SU(2)}} + \hat{w}^2 + \frac{1}{16}\lambda^2\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}\psi^2\right) \,, \tag{2.44}$$

[9] Therefore (2.43) has the geometrical interpretation that the transverse six-dimensional space $P^{\perp}$ is conformally balanced.

with corresponding differential conditions

$$\begin{aligned}
\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3\rho^2}\,\hat{w}\right) &= 2\lambda^{-1/2}\hat{J}_3 + 2\rho\lambda\hat{w}\wedge\hat{\rho} \ , \\
\mathrm{d}\left(\lambda^{-3/2}\hat{J}_1\wedge\hat{w} - \rho\hat{J}_2\wedge\hat{\rho}\right) &= 0 \ , \\
\mathrm{d}\left(\hat{J}_2\wedge\hat{w} + \lambda^{-3/2}\rho^{-1}\hat{J}_1\wedge\hat{\rho}\right) &= 0 \ .
\end{aligned} \tag{2.45}$$

The flux $F$ in (2.39) then becomes

$$F = \frac{1}{4}\mathrm{d}\psi\wedge\mathrm{d}\left(\lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}\hat{J}_1\right) \ . \tag{2.46}$$

These expressions precisely coincide with those in section 7.2 of [42]. Of course, this is an important cross-check of our general formulae.

Notice that the Bianchi identity for $F$ is satisfied automatically from the expression in (2.46). In fact for the general $m = 0$ class of geometries the Bianchi identity and equation of motion for $F$ read

$$\mathrm{d}F = 0 \ , \qquad \mathrm{d}\left(\mathrm{e}^{3\Delta}\star F\right) = 0 \ . \tag{2.47}$$

Defining the conformally related metric $\tilde{g}_7 = \mathrm{e}^{-6\Delta}g_7$, the equation of motion for $F$ becomes $\mathrm{d}\,\tilde{\star}F = 0$. It follows that $F$ is a harmonic four-form on $(Y_7, \tilde{g})$. In particular, imposing also flux quantization we see that $F$ defines a non-trivial cohomology class in $H^4(Y_7;\mathbb{Z})$, which we may associate with the M5-brane charge of the solution.

When $m = 0$ there is no "electric" component of the four-form flux $G$, and these AdS$_4$ backgrounds have the physical interpretation of being created by wrapped M5-branes. Indeed, as we shall see in section 2.3, when $m \neq 0$ there is always a non-zero quantized M2-brane charge $N \in \mathbb{N}$, with the supergravity description being valid in a large $N$ limit. The supergravity free energy then scales universally as $N^{3/2}$. One would expect the free energy of the M5-brane solutions, sourced by the internal "magnetic" flux $F$, to scale as $N^3$, where the cohomology class in $H^4(Y_7;\mathbb{Z})$ defined by $F$ scales as $N$. However, the lack of a contact structure in this case (see below) means that a proof would look rather different from the analysis in section 2.3.

In section 9.5 of [42] the authors found a solution within the $m = 0$ class, solving the system (2.45), describing the near-horizon limit of M5-branes wrapping a Special Lagrangian three-cycle $\Sigma_3$. In fact this is the eleven-dimensional uplift of a seven-dimensional solution found originally in reference [47]. The internal seven-manifold $Y_7$ takes the form of an $S^4$ fibration over $\Sigma_3$, where the latter is endowed with an Einstein metric of constant negative curvature. As one sees explicitly from the solution, the R-symmetry vector field $\partial_\psi$ acts on $S^4 \subset \mathbb{R}^5 = \mathbb{R}^3 \oplus \mathbb{R}^2$ by rotating the $\mathbb{R}^2$ factor in the latter decomposition. In particular, there is a fixed copy of $S^2$, implying that $\partial_\psi$ does *not* define a one-dimensional foliation in this $m = 0$ case. Notice this also implies there cannot be any compatible global contact structure, again in contrast with the $m \neq 0$ geometries. The flux $F$ generates the cohomology group $H^4(\Sigma_3 \times S^4; \mathbb{R}) \cong \mathbb{R}$.

As far as we are aware, the solution in section 9.5 of [42] is the only known solution in this class. It would certainly be very interesting to know if there are more AdS$_4$ geometries sourced only by M5-branes.

### 2.2.8 Reduction of the equations in components

In this section we further analyze the system of supersymmetry equations (2.36), extracting information from each component under the natural $1+1+1+4$ decomposition implied by the three-frame (2.32). Since we have dealt with the $m=0$ equations in the previous section, we henceforth take $m \neq 0$ in the remainder of the paper.

We begin by defining the one-form

$$\mathscr{B} \equiv \frac{36}{m^2} \mathrm{e}^{6\Delta} \left\|\xi\right\|^2 (\mathrm{d}\tau + \mathscr{A}) \ , \tag{2.48}$$

which appears in the frame element $e^6$ in (3.14), so that

$$e^6 = \frac{m\mathrm{e}^{-3\Delta}|S|}{24\left\|\xi\right\|\sqrt{1-\left\|\xi\right\|^2}} \mathscr{B} \ , \tag{2.49}$$

and further decompose

$$\mathscr{B} \equiv \mathscr{B}_\tau \mathrm{d}\tau + \hat{\mathscr{B}} \ , \tag{2.50}$$

where $\partial_\tau \lrcorner \hat{\mathscr{B}} = 0$. Since also $e^7$ and $e^5$ are orthogonal to $\mathscr{B}$, it follows that $\hat{\mathscr{B}}$ is a linear combination of $e^a$, $a = 1,2,3,4$, the orthonormal frame for the four-metric $g_{SU(2)}$ in (2.34). It is also convenient to rescale the latter four-metric, together with its $SU(2)$ structure, via

$$\hat{J}_I \equiv \frac{24}{m} \mathrm{e}^{3\Delta} J_I \ , \qquad I = 1,2,3 \ , \tag{2.51}$$

so that correspondingly $\widehat{g_{SU(2)}} = \frac{24}{m}\mathrm{e}^{3\Delta} g_{SU(2)}$.[10] Notice this makes sense only when $m \neq 0$.

Given the coordinates $(\psi, \tau, \rho)$ defined via (3.14), it is then natural to decompose the exterior derivative as

$$\mathrm{d} = \mathrm{d}\psi \wedge \frac{\partial}{\partial\psi} + \mathrm{d}\tau \wedge \frac{\partial}{\partial\tau} + \mathrm{d}\rho \wedge \frac{\partial}{\partial\rho} + \hat{\mathrm{d}} \ , \tag{2.52}$$

where from now on hatted expressions will (essentially) denote four-dimensional quantities. We may then decompose the exterior derivatives and forms in the supersymmetry

[10] This scaling is different from the scaling used in section 2.2.7, where $m=0$.

equations (2.36) under this natural $1+1+1+4$ splitting.

Beginning with the first equation in (2.36), the utility of the definition (2.48) is that this first supersymmetry equation becomes simply

$$\mathrm{d}\mathscr{B} \;=\; 2\hat{J}_3 - \frac{1}{2}\omega\rho\,\mathrm{d}\rho\wedge\mathscr{B}\ . \tag{2.53}$$

where to simplify resulting equations it is useful to define the function

$$\omega \;\equiv\; \frac{\mathrm{e}^{-6\Delta}}{1-\|\xi\|^2}\ . \tag{2.54}$$

Decomposing as outlined above, this becomes

$$\begin{aligned}\partial_\tau\hat{\mathscr{B}} &= \hat{\mathrm{d}}\mathscr{B}_\tau\ ,\\ \partial_\rho\mathscr{B} &= -\frac{1}{2}\omega\rho\,\mathscr{B}\ ,\\ \hat{\mathrm{d}}\hat{\mathscr{B}} &= 2\hat{J}_3\ .\end{aligned} \tag{2.55}$$

Note here that everything is invariant under $\partial_\psi$. The integrability condition for (2.53) immediately implies that $\partial_\tau\hat{J}_3 = 0 = \hat{\mathrm{d}}\hat{J}_3$, while combining the component

$$\hat{\mathrm{d}}\left(\omega\mathscr{B}_\tau\right) - \partial_\tau\left(\omega\hat{\mathscr{B}}\right) \;=\; 0\ , \tag{2.56}$$

with the first and last equation in (2.55) leads to the conclusion

$$\partial_\tau\Delta \;=\; 0 \;=\; \hat{\mathrm{d}}\Delta\ , \tag{2.57}$$

so that the warp factor $\Delta$, and the related functions $|S|$ and $\xi$, all depend only on the coordinate $\rho$!

The other two equations in (2.36) may be analyzed similarly. Rather than present all the details, which are straightforward but rather long, we simply present the final result. Defining $\hat{\Omega} = \hat{J}_1 + \mathrm{i}\hat{J}_2$, the supersymmetry conditions (2.36) are equivalent to the equations

$$\boxed{\begin{array}{lclclcl}\hat{\mathrm{d}}\hat{\mathscr{B}} &=& 2\hat{J}_3 & \hat{\mathrm{d}}\hat{\Omega} &=& \left(\upsilon-\mathrm{i}u\hat{\Omega}\right)\mathscr{B}_\tau^{-1}\wedge\hat{\mathscr{B}}\\ \partial_\rho\mathscr{B} &=& -\frac{1}{2}\omega\rho\,\mathscr{B} & \left[\partial_\rho\hat{\Omega}\right]_+ &=& -\frac{1}{2}\omega\rho\,\hat{\Omega}\\ \partial_\tau\hat{\mathscr{B}} &=& \hat{\mathrm{d}}\mathscr{B}_\tau & \partial_\tau\hat{\Omega} &=& -\mathrm{i}u\hat{\Omega}+\upsilon\end{array}} \tag{2.58}$$

Here we have defined the two-form

$$\upsilon \;\equiv\; \frac{m}{6}\mathrm{e}^{-3\Delta}\left(\left[\rho\,\partial_\rho\hat{J}_2\right]_- - \frac{\mathrm{i}}{\|\xi\|^2}\left[\rho\,\partial_\rho\hat{J}_1\right]_-\right)\mathscr{B}_\tau\ , \tag{2.59}$$

and the function

$$u \equiv \frac{m}{6}\mathrm{e}^{-3\Delta}\left(\frac{1}{2}\rho\,\partial_\rho \log\omega - \rho^2\omega\right)\mathscr{B}_\tau\ . \tag{2.60}$$

The notation $[\,\cdot\,]_\pm$ denotes the self-dual and anti-self-dual parts of a two-form along the four-dimensional $SU(2)$-structure space. In particular, of course $\hat{J}_I$, $I=1,2,3$, form a basis for the self-dual forms. We also note that the integrability condition for the first three equations in (2.58) gives

$$\partial_\tau\hat{J}_3 = 0\ , \qquad \partial_\rho\hat{J}_3 = -\frac{1}{2}\omega\rho\,\hat{J}_3\ , \qquad \hat{\mathrm{d}}\hat{J}_3 = 0\ . \tag{2.61}$$

As an aside comment, we notice that a subset of the equations in (2.58) may be re-interpreted as equations for a dynamical *contact-hypo* structure on a five-dimensional space [48, 49]. Here we decompose the seven-dimensional manifold under a $1+1+5$ split, where the two transverse directions are parametrized by the coordinates $\rho$ and $\psi$. The $(\mathscr{B}, J_I)$ then define a contact-hypo structure (at fixed $\rho$) obeying the equations

$$\tilde{\mathrm{d}}\mathscr{B} = 2\hat{J}_3\ , \qquad \tilde{\mathrm{d}}\hat{\Omega} = \left(\upsilon - \mathrm{i}u\hat{\Omega}\right)\mathscr{B}_\tau^{-1}\wedge\mathscr{B}\ , \tag{2.62}$$

where $\tilde{\mathrm{d}} \equiv \mathrm{d}\tau\wedge\frac{\partial}{\partial\tau} + \hat{\mathrm{d}}$. Note that when $\left[\partial_\tau\hat{\Omega}\right]_- = 0$ these become the conditions characterizing a Sasaki-Einstein five-manifold. However, in this paper we will not pursue further this point of view.

We emphasize again that since $\Delta$ is a function only of $\rho$, this implies that the derived functions $|S|$ and $\xi$ also depend only on $\rho$. We conclude by writing an even more explicit expression for the flux given in (2.39):

$$\boxed{\begin{aligned} F \;=\; & -\tfrac{1}{\|\xi\|}\left(12\mathrm{e}^{6\Delta}\left\|\xi\right\|^2\partial_\rho\Delta - 6\rho\right)e^{57}\wedge J_1 + 12\mathrm{e}^{6\Delta}\partial_\rho\Delta e^{67}\wedge J_2 \\ & - m\tfrac{\sqrt{1-\|\xi\|^2}}{\|\xi\|}e^{56}\wedge J_1 - \tfrac{m}{6}\mathrm{e}^{3\Delta}(1-\left\|\xi\right\|^2)e^{67}\wedge\left[\partial_\rho J_2\right]_- \\ & - \tfrac{m}{6}\mathrm{e}^{3\Delta}\tfrac{(1-\|\xi\|^2)}{\|\xi\|}e^{57}\wedge\left[\partial_\rho\hat{J}_1\right]_-\ . \end{aligned}} \tag{2.63}$$

This expression is particularly useful for proving sufficiency of the differential system in subsection 2.2.6.

We shall investigate the general equations (2.58), in a special case, in section 2.4, reducing them to a single second order ODE in $\rho$.

## 2.3 M2-brane solutions

In this section we further elaborate on the geometry and physics of solutions with $m\neq 0$. In particular we show that all such solutions admit a canonical contact structure, for which the R-symmetry Killing vector $\xi$ is the Reeb vector field. Many physical properties of the solutions, such as the free energy and scaling dimensions of BPS wrapped M5-branes, can be expressed purely in terms of this contact structure.

### 2.3.1 Contact structure

When $m \neq 0$ we may define a one-form $\sigma$ via

$$P \equiv \frac{m}{6}\mathrm{e}^{-3\Delta}\sigma , \tag{2.64}$$

where $P$ is the one-form bilinear defined in the second line in (2.17). In terms of our frame (3.14), we then have

$$\begin{aligned} \sigma &= \frac{1}{\|\xi\|}e^7 + \frac{6\mathrm{e}^{3\Delta}}{m}\frac{|S|\sqrt{1-\|\xi\|^2}}{\|\xi\|}e^6 , \\ &= \frac{1}{4}\left[\mathrm{d}\psi + \mathscr{A} + \left(\tfrac{6}{m}\right)^2\rho^2(\mathrm{d}\tau + \mathscr{A})\right] . \end{aligned} \tag{2.65}$$

Up to a factor of $m/6$, the one-form inside the square bracket on the left hand side of the first equation in (2.36) is in fact $\sigma$. Thus we read off

$$\mathrm{d}\sigma = \frac{12}{m}\mathrm{e}^{3\Delta}\left(J_3 - \|\xi\|\, e^5\wedge e^6\right) , \tag{2.66}$$

and a simple algebraic computation then leads to

$$\sigma\wedge(\mathrm{d}\sigma)^3 = \frac{2^7 3^4}{m^3}\mathrm{e}^{9\Delta}\mathrm{vol}_7 . \tag{2.67}$$

Here

$$\mathrm{vol}_7 \equiv -e^5\wedge e^6\wedge e^7\wedge \mathrm{vol}_4 = -\frac{1}{2}e^5\wedge e^6\wedge e^7\wedge J_3\wedge J_3 , \tag{2.68}$$

denotes the Riemannian volume form of $Y_7$ (with a convenient choice of orientation). It follows that when $m\neq 0$, the seven-form $\sigma\wedge(\mathrm{d}\sigma)^3$ is a nowhere-zero top degree form on $Y_7$, and thus by definition $\sigma$ is a contact form on $Y_7$.

Again, straightforward algebraic computations using the Fierz identity in appendix A lead to

$$\xi\lrcorner\sigma = 1 , \qquad \xi\lrcorner\mathrm{d}\sigma = 0 . \tag{2.69}$$

This implies that the Killing vector field $\xi$ is also the unique Reeb vector field for the contact structure defined by $\sigma$.

### 2.3.2 Flux quantization

When $m \neq 0$, equation (2.27) immediately leads to the natural gauge choice

$$F \;=\; \mathrm{d}A \;, \tag{2.70}$$

where $A$ is the global three-form

$$A \;\equiv\; \frac{6}{m}\mathrm{e}^{6\Delta}\mathrm{Im}\,\bar{\chi}_+^c\gamma_{(3)}\chi_- \;. \tag{2.71}$$

In terms of our frame, this reads

$$A \;=\; \frac{6}{m}\mathrm{e}^{6\Delta}\Big[|S|J_2\wedge e^5 - \frac{1}{\|\xi\|}J_1\wedge\Big(|S|e^6 + \frac{m}{6}\mathrm{e}^{-3\Delta}\sqrt{1-\|\xi\|^2}e^7\Big)\Big] \;. \tag{2.72}$$

Notice that, either using the last expression or using (2.23), we find that

$$\mathcal{L}_\xi A \;=\; 0 \;. \tag{2.73}$$

Of course, one is free to add to $A$ any closed three-form $a$, which will result in the same curvature $F$

$$A \;\to\; A + \frac{1}{(2\pi\ell_p)^3}a \;. \tag{2.74}$$

If $a$ is exact this is a gauge transformation of $A$ and leads to a physically equivalent M-theory background. In fact more generally if $a$ has integer periods then the transformation (2.74) is a large gauge transformation of $A$, again leading to an equivalent solution. It follows that only the cohomology class of $a$ in the torus $H^3(Y_7;\mathbb{R})/H^3(Y_7;\mathbb{Z})$ is a physically meaningful parameter, and this corresponds to a marginal parameter in the dual CFT. In fact the free energy will be independent of this choice of $a$, which is why we have set $a = 0$ in (2.71). There is also the possibility of adding discrete torsion to $A$ when $H^4_{\mathrm{torsion}}(Y_7;\mathbb{Z})$ is non-trivial, but we will not discuss this here.

The flux quantization condition in eleven dimensions is

$$N \;=\; -\frac{1}{(2\pi\ell_p)^6}\int_{Y_7}\Big(*_{11}G + \frac{1}{2}C\wedge G\Big) \;, \tag{2.75}$$

where $N$ is the total M2-brane charge. Dirac quantization requires that $N$ is an integer. Substituting our ansatz (2.3) into (2.75) leads to

$$N \;=\; \frac{1}{(2\pi\ell_p)^6}\int_{Y_7}\Big(m\mathrm{e}^{3\Delta}\mathrm{vol}_7 - \frac{1}{2}A\wedge F\Big) \;, \tag{2.76}$$

where $\mathrm{vol}_7$ denotes the Riemannian volume form for $Y_7$. By far the simplest way to evaluate $A\wedge F$ is to use the identity (A.1) with $\mathcal{C} = 1$. Using (2.71), this immediately leads to an expression for $A\wedge F$ in terms of vol$_7$, and using (2.67) we obtain

$$N \;=\; \frac{1}{(2\pi\ell_p)^6}\frac{m^2}{2^5 3^2}\int_{Y_7}\sigma\wedge(\mathrm{d}\sigma)^3\ . \tag{2.77}$$

In particular, we see that $m\neq 0$ leads to a non-zero M2-brane charge $N$.

### 2.3.3 The free energy

The effective four-dimensional Newton constant $G_4$ is computed by dimensional reduction of eleven-dimensional supergravity on $Y_7$. More precisely, by definition $1/16\pi G_4$ is the coefficient of the four-dimensional Einstein-Hilbert term, in Einstein frame. A standard computation leads to the formula

$$\frac{1}{16\pi G_4} \;=\; \frac{\pi\int_{Y_7}\mathrm{e}^{9\Delta}\mathrm{vol}_7}{2(2\pi\ell_p)^9}\ . \tag{2.78}$$

On the other hand, $G_4$ also determines the gravitational free energy $\mathcal{F}_{\mathrm{AdS}}$

$$\mathcal{F}_{\mathrm{AdS}} \;\equiv\; -\log|Z| \;=\; \frac{\pi}{2G_4}\ . \tag{2.79}$$

Here the left hand side of (2.79) is the free energy of the unit radius AdS$_4$ computed in Euclidean quantum gravity, where $Z$ is the gravitational partition function. Thus in the supergravity approximation, $\mathcal{F}_{\mathrm{AdS}}$ is simply the four-dimensional on-shell Einstein-Hilbert action, which has been regularized to give the finite result on the right hand side of (2.79) using the boundary counterterm subtraction method of [50, 51]. Via the AdS/CFT correspondence, $\mathcal{F}_{\mathrm{AdS}}=\mathcal{F}_{\mathrm{CFT}}\equiv\mathcal{F}$, where $\mathcal{F}_{\mathrm{CFT}}$ is the free energy of the dual CFT on the conformal boundary $S^3$ of AdS$_4$. Combining (2.78) and (2.79) then leads to the supergravity formula

$$\mathcal{F} \;=\; \frac{4\pi^3\int_{Y_7}\mathrm{e}^{9\Delta}\mathrm{vol}_7}{(2\pi\ell_p)^9}\ . \tag{2.80}$$

Combining (2.77), (2.80) and (2.67) leads to our final formula

$$\boxed{\mathcal{F} \;=\; N^{3/2}\sqrt{\frac{32\pi^6}{9\int_{Y_7}\sigma\wedge(\mathrm{d}\sigma)^3}}\ .} \tag{2.81}$$

We see that the famous $N^{3/2}$ scaling behaviour of the free energy of $N$ M2-branes continues to hold in the most general $\mathcal{N}=2$ supersymmetric case with flux turned on. Moreover, the coefficient is expressed purely in terms of the contact volume of $Y_7$. In the Sasaki-Einstein case this agrees with the Riemannian volume computed using vol$_7$, but more generally the two volumes are different. The contact volume has the property, in the sense described precisely in appendix B of [52], that it depends only on the Reeb vector field $\xi$ determined by the contact structure. In particular, if we formally consider varying the contact structure of a given solution, the contact volume is a strictly convex function of the Reeb vector field $\xi$. It is of course natural to conjecture that this function is related as in (2.79) to minus the logarithm of the field theoretic $|Z|$-function defined in [33], as a function of a trial R-symmetry in the dual supersymmetric field theory on $S^3$. This was conjectured in the Sasaki-Einstein case in [23], and has by now been verified in a large number of examples, including infinite families [26]. The contact volume has the desirable property that it can be computed using topological and fixed point theorem methods, so that one can compute the free energy of a solution essentially knowing only its Reeb vector field. We will illustrate this with the class of solutions in section 2.4.

Finally, the scaling symmetry of eleven-dimensional supergravity in which the metric $g_{11}$ and four-form $G$ have weights two and three, respectively, leads to a symmetry in which one shifts $\Delta \to \Delta + c$ and simultaneously scales $m \to \mathrm{e}^{3c} m$, $F \to \mathrm{e}^{3c} F$, where $c$ is any real constant. We may then take the metric on $Y_7$ to be of order $\mathcal{O}(N^0)$, and conclude from the quantization condition (2.75), which has weight 6 on the right hand side, that $\mathrm{e}^{\Delta} = \mathcal{O}(N^{1/6})$. It follows that the AdS$_4$ radius, while dependent on $Y_7$, is $R_{\mathrm{AdS}_4} = \mathrm{e}^{\Delta} = \mathcal{O}(N^{1/6})$, and that the supergravity approximation we have been using is valid only in the $N \to \infty$ limit.

### 2.3.4 Scaling dimensions of BPS wrapped M5-branes

A probe M5-brane whose world-space is wrapped on a generalized calibrated five-submanifold $\Sigma_5 \subset Y_7$ and which moves along a geodesic in AdS$_4$ is expected to correspond to a BPS operator $\mathcal{O}_{\Sigma_5}$ in the dual three-dimensional SCFT. In particular, when $Y_7$ is a Sasaki-Einstein manifold, the scaling dimension of this operator can be calculated from the volume of the five-submanifold $\Sigma_5$ [53]. In this section we show that a simple generalization of this correspondence holds for the general $\mathcal{N} = 2$ supersymmetric AdS$_4 \times Y_7$ solutions treated in this paper.[11]

Given a Killing spinor $\epsilon$ of eleven-dimensional supergravity, it is simple to derive the following BPS bound for the M5-brane [8, 56]

$$\epsilon^\dagger \epsilon \, L_{\mathrm{DBI}} \, \mathrm{vol}_5 \quad \geq \quad \left[ \frac{1}{2} (j^* k \lrcorner H) \wedge H + j^* \mu \wedge H + j^* \nu \right] . \qquad (2.82)$$

Here $H$ is the three-form on the M5-brane, defined by $H = h + j^* C$ where $h$ is closed and $j^*$ denotes the pull-back to the M5-brane world-volume. The one-form $k$, two-form $\mu$ and five-form $\nu$ are defined [43] by the eleven-dimensional bilinears

$$k \;\equiv\; \bar{\epsilon} \Gamma_{(1)} \epsilon \; , \qquad \mu \;\equiv\; \bar{\epsilon} \Gamma_{(2)} \epsilon \; , \qquad \nu \;\equiv\; \bar{\epsilon} \Gamma_{(5)} \epsilon \; , \qquad (2.83)$$

[11] Such supersymmetric M5-branes exist only for certain boundary conditions [54, 55], and our discussion here applies to these cases.

and $\mathrm{vol}_5$ is the volume form on the world-space of the M5-brane. We have defined $\bar{\epsilon} \equiv \epsilon^\dagger \Gamma_0$ as usual.

The bound (2.82) follows from the inequality

$$\left\|\mathcal{P}_-\epsilon\right\|^2 = \epsilon^\dagger \mathcal{P}_-\epsilon \geq 0 , \tag{2.84}$$

where $\mathcal{P}_- \equiv (1-\tilde{\Gamma})/2$ is the $\kappa$-symmetry projector and $\tilde{\Gamma}$ is the traceless Hermitian product structure

$$\tilde{\Gamma} \equiv \frac{1}{L_{\mathrm{DBI}}}\Gamma_0\left[\frac{1}{4}(j^*\Gamma)^a(H^*\lrcorner H)_a + \frac{1}{2!}(j^*\Gamma)^{a_1a_2}H^*_{a_1a_2} + \frac{1}{5!}(j^*\Gamma)^{a_1\cdots a_5}\varepsilon_{a_1\cdots a_5}\right] . \tag{2.85}$$

Here $a, a_1 \ldots a_5 = 1,\ldots,5$, where the two-form $H^* \equiv *_5 H$ is the world-space dual of $H$. This bound is saturated if and only if $\mathcal{P}_-\epsilon = 0$ and corresponds to a probe M5-brane preserving supersymmetry.

We write the AdS$_4$ metric in global coordinates (*cf.* footnote 1) and choose the static gauge embedding $\{t=\sigma^0, x^m = \sigma^m\}$, where $t$ is global time in AdS$_4$ and $x^m$, with $m=1,\ldots,5$, are coordinates on $Y_7$. The Dirac-Born-Infeld Lagrangian $L_{\mathrm{DBI}}$ is then defined by $L_{\mathrm{DBI}} = \sqrt{\det(\delta_m^n + H_m^{*n})}$. The vector $k_\sharp$ dual to the one-form $k$ is a time-like Killing vector, which using the explicit form of the eleven-dimensional $\mathcal{N}=2$ Killing spinor (2.9), and an appropriate choice of AdS$_4$ spinors $\psi_i$, reads

$$k_\sharp = \partial_t + \frac{1}{2}\xi . \tag{2.86}$$

Accordingly, $\epsilon^\dagger\epsilon = k_\sharp^0 = \frac{1}{2}\mathrm{e}^\Delta \cosh\varrho$, and hence the bound (2.82) is saturated when $\varrho = 0$ (*i.e.* the M5-brane is at the centre of AdS$_4$) and

$$\frac{\mathrm{e}^\Delta}{2}L_{\mathrm{DBI}}\,\mathrm{vol}_5 = \left[\frac{1}{2}(j^*k\lrcorner H)\wedge H + j^*\mu\wedge H + j^*\nu\right] . \tag{2.87}$$

The energy density of an M5-brane can be computed by solving the Hamiltonian constraints [8, 56]. For the static gauge embedding and $\varrho = 0$ these lead to

$$\mathcal{E} = P_t = T_{\mathrm{M5}}\left(\frac{\mathrm{e}^\Delta}{2}L_{\mathrm{DBI}} + \mathcal{C}_t\right) , \tag{2.88}$$

where $T_{\mathrm{M5}} = 2\pi/(2\pi\ell_p)^6$ is the M5-brane tension and the contribution from the Wess-Zumino coupling is $\mathcal{C}_t\mathrm{vol}_5 = \partial_t\lrcorner C_6 - \frac{1}{2}(\partial_t\lrcorner C)\wedge(C-2H)$, with the potential $C_6$ defined through $\mathrm{d}C_6 = *_{11}G + \frac{1}{2}C\wedge G$. However, from the explicit expression of $C$ one can check that we have $\mathcal{C}_t = 0$. The M5-brane energy is then given by

$$E_{\mathrm{M5}} = T_{\mathrm{M5}}\int_{\Sigma_5}\frac{\mathrm{e}^\Delta}{2}L_{\mathrm{DBI}}\,\mathrm{vol}_5 = T_{\mathrm{M5}}\int_{\Sigma_5}\left(\frac{1}{4}(\xi\lrcorner H)\wedge H + j^*\mu\wedge H + j^*\nu\right) , \tag{2.89}$$

where we used (2.86). Let us briefly discuss this expression for the energy. With our gauge choice (2.71) for the three-form potential, in general we have $H = A + h$, where $h$ is a closed three-form. If $h$ is exact and invariant[12] under $k_\sharp$, namely $h = \mathrm{d}b$ with $\mathcal{L}_{k_\sharp} b = 0$, then one can check that the integral does not depend on $h$. To see this, one has to recall that $\mathcal{L}_{k_\sharp} A = 0$, use the results of [43], and apply Stokes' theorem repeatedly. If $h$ is not exact, *a priori* it will contribute to the energy, and hence we expect the dimension of the dual operator to be affected. We leave an investigation of this interesting possibility for future work, and henceforth set $H = A$. In particular, $A$ is expressed as a bilinear of $\chi_\pm$ in (2.71).

Using the explicit form of the eleven-dimensional $\mathcal{N} = 2$ Killing spinor (2.9) and the static gauge embedding one derives

$$\begin{aligned} \iota^* k &= \tfrac{1}{2} \mathrm{e}^{2\Delta} K \ , \\ \iota^* \mu &= 4\mathrm{e}^{3\Delta} \left\{ -\tfrac{1}{8} \mathrm{Im} \, [\bar{\chi}_+^c \gamma_{(2)} \chi_-] + \mathrm{Im}[\bar{\psi}_1^+ (\psi_2^+)^c] \mathrm{Re} \, [\bar{\chi}_+^c \gamma_{(2)} \chi_-] \right\} \ , \\ \iota^* \nu &= 4\mathrm{e}^{6\Delta} \star \left\{ \tfrac{1}{8} \mathrm{Re} \, [\bar{\chi}_+^c \gamma_{(2)} \chi_-] + \mathrm{Im}[\bar{\psi}_1^+ (\psi_2^+)^c] \mathrm{Im} \, [\bar{\chi}_+^c \gamma_{(2)} \chi_-] \right\} \ , \end{aligned} \tag{2.90}$$

where $\iota^*$ denotes a pull-back to $Y_7$, and where the constant scalar bilinear $\mathrm{Re}[\bar{\psi}_1^+ (\psi_2^+)^c]$ is rescaled for convenience to $\frac{1}{8}$. The $\chi_\pm$ bilinears can then be expressed in terms of $e^5, e^6, e^7$ and $J_I$. The non-constant scalar $\mathrm{Im}[\bar{\psi}_1^+ (\psi_2^+)^c]$ drops out of the calculation and one arrives at[13]

$$\frac{1}{2} (j^* k \lrcorner H) \wedge H + j^* \mu \wedge H + j^* \nu \ = \ -\frac{m^2}{2^6 3^2} \sigma \wedge (\mathrm{d}\sigma)^2 \ . \tag{2.91}$$

Hence we get the remarkably simple result

$$E_{\mathrm{M5}} \ = \ -T_{\mathrm{M5}} \frac{m^2}{2^6 3^2} \int_{\Sigma_5} \sigma \wedge (\mathrm{d}\sigma)^2 \ . \tag{2.92}$$

Combining the latter with (2.77), and identifying $\Delta(\mathcal{O}_{\Sigma_5})$ with the energy $E_{\mathrm{M5}}$ in global AdS, leads straightforwardly to the formula

$$\boxed{\Delta(\mathcal{O}_{\Sigma_5}) \ = \ \pi N \left| \frac{\int_{\Sigma_5} \sigma \wedge (\mathrm{d}\sigma)^2}{\int_{Y_7} \sigma \wedge (\mathrm{d}\sigma)^3} \right| \ .} \tag{2.93}$$

The scaling dimensions of operators dual to BPS wrapped M5-branes are thus also determined purely by the contact structure. As for the contact volume of $Y_7$, the right hand side of (2.93) can again be computed from a knowledge of $\Sigma_5$ and the Reeb vector field $\xi$.

[12] One should obviously require that $\partial_t$ and $\xi$ generate symmetries of the M5-brane action.
[13] The sign arises from our choice of conventions, *cf.* [45].

## 2.4 Special class of solutions: $\partial_\tau$ Killing

Since the general system of supersymmetry equations presented in section 2.2.8 is rather complicated, in this section we impose a single simplifying assumption, namely that $\partial_\tau$ is a *Killing vector* field for the metric[14] $g_7$. There are two motivations for this. Firstly, it is clearly a natural geometric condition. Secondly, the only solution in the literature in the $m \neq 0$ class that is not Sasaki-Einstein is the Corrado-Pilch-Warner solution [27]. This solution describes the infrared fixed point of a massive deformation of the maximally supersymmetric AdS$_4 \times S^7$ solution, and has the same topology but with non-standard metric on $S^7$ and flux. We will first show that the assumption that $\partial_\tau$ is Killing immediately leads to the four-metric $g_{SU(2)}$ being conformal to a Kähler-Einstein metric, and that the supersymmetry conditions then entirely reduce to a single second order non-linear ODE. The Corrado-Pilch-Warner solution is a particular solution to this ODE, with $g_{SU(2)}$ being (conformal to) the standard Fubini-Study metric on $\mathbb{CP}^2$. We will then show numerically that there exists a second solution, dual to the infrared fixed point of a cubic deformation of $N$ M2-branes at a general CY$_3 \times \mathbb{C}$ singularity, where CY$_3$ denotes any Calabi-Yau three-fold cone. In particular, when CY$_3 = \mathbb{C}^3$ endowed with a flat metric, this leads to a new, smooth $\mathcal{N}=2$ supersymmetric AdS$_4 \times S^7$ solution.

### 2.4.1 Further reduction of the equations

Let us analyze the conditions (2.58), with the assumption that $\partial_\tau$ is Killing. Notice that the latter implies

$$\left[\partial_\tau \hat{J}_I\right]_\pm \;=\; \partial_\tau\left[\hat{J}_I\right]_\pm \;=\; \begin{cases} \partial_\tau \hat{J}_I \\ 0 \end{cases} . \tag{2.94}$$

The left hand side of last equation in (2.58) is thus identically zero. Taking the real and imaginary parts of the right hand side then implies that $\partial_\rho\hat{\Omega}$ is self-dual. The plus subscripts may then be dropped in the second line of (2.58), and we see that

$$\partial_\rho \hat{J}_I \;=\; -\frac{1}{2}\omega\rho\hat{J}_I \,, \tag{2.95}$$

holds for all $I=1,2,3$. Recalling that $\Delta$ is always a function only of $\rho$, we may introduce the rescaled $SU(2)$ structure

$$\hat{J}_I \;\equiv\; f_1(\rho)\mathbb{J}_I \,, \qquad I=1,2,3 \,, \tag{2.96}$$

[14] Note that we are not requiring that $\partial_\tau$ generates a symmetry of the full solution. Indeed we will show that in general the flux $F$ is *not* invariant under $\partial_\tau$.

and see that provided $f_1(\rho)$ satisfies the differential equation

$$\frac{\mathrm{d}f_1}{\mathrm{d}\rho} \quad = \quad -\frac{1}{2}\omega\rho f_1 \ , \tag{2.97}$$

then the $SU(2)$-structure two-forms $\mathbb{J}_I$ are independent of $\rho$.

Similarly, the Killing condition on $\partial_\tau$ implies that $\mathscr{B}_\tau$ and $\hat{\mathscr{B}}$ are independent of $\tau$, and the first equation in (2.58) then implies that $\mathscr{B}_\tau = \mathscr{B}_\tau(\rho)$ depends only on $\rho$. We may then similarly solve the second equation in (2.58) by rescaling

$$\mathscr{B} \quad \equiv \quad f_1(\rho)\mathbb{B} \ , \tag{2.98}$$

and deduce that $\mathbb{B}$ is independent of both $\tau$ and $\rho$. Similarly writing

$$\mathbb{B} \quad \equiv \quad \mathbb{B}_\tau \mathrm{d}\tau + \hat{\mathbb{B}} \ , \tag{2.99}$$

where now $\mathbb{B}_\tau$ is a constant, the remaining equations in (2.58) are

$$\begin{aligned} \hat{\mathrm{d}}\hat{\mathbb{B}} \quad &= \quad 2\mathbb{J}_3 \ , \qquad \hat{\mathrm{d}}(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \ = \ -\mathrm{i}f_1 u \mathscr{B}_\tau^{-1}(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2)\wedge\hat{\mathbb{B}} \ , \\ \partial_\tau(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \quad &= \quad -\mathrm{i}u(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \ . \end{aligned} \tag{2.100}$$

Since the blackboard script quantities are independent of $\rho$, the second equation in (2.100) implies that

$$\frac{m}{6}\mathrm{e}^{-3\Delta}\left(\frac{1}{2}\rho\,\partial_\rho \log\omega - \rho^2\omega\right) f_1 = -\gamma \tag{2.101}$$

which is *a priori* a function of $\rho$, is in fact a constant. In order to remove the explicit factors of $m$, and write everything in terms of a single function, it is convenient to rescale

$$r \quad \equiv \quad \frac{6}{m}\rho \ , \qquad f_2^2(r) \ \equiv \ \left(\frac{m}{6}\right)^2 \omega \ . \tag{2.102}$$

In terms of these new variables, the differential equations (2.97), (2.101) read

$$\boxed{\begin{aligned} f_1' \quad &= \quad -\tfrac{1}{2} r f_1 f_2^2 \ , \\ f_2' \quad &= \quad -\gamma r^{-1}\sqrt{1 + f_2^2(1+r^2)} + r f_1 f_2^3 \ , \end{aligned}} \tag{2.103}$$

which are a coupled set of first order ODEs for the functions $f_1(r)$, $f_2(r)$, and from henceforth a prime will denote derivative with respect to the coordinate $r$. The remaining supersymmetry conditions (2.100) now simplify to

$$\begin{aligned} \hat{\mathrm{d}}\hat{\mathbb{B}} \quad &= \quad 2\mathbb{J}_3 \ , \qquad \hat{\mathrm{d}}(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \ = \ \mathrm{i}\gamma(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2)\wedge\hat{\mathbb{B}} \ , \\ \partial_\tau(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \quad &= \quad \mathrm{i}\gamma\mathbb{B}_\tau(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) \ . \end{aligned} \tag{2.104}$$

Here both $\gamma$ and $\mathbb{B}_\tau$ are constants. The first line says that the four-metric defined by $(\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3)$ is Kähler-Einstein with Ricci tensor satisfying Ric $= 2\gamma g_{\mathrm{KE}}$. The second equation is solved simply by multiplying $\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2$ by a phase $\mathrm{e}^{-\mathrm{i}\gamma\mathbb{B}_\tau}$, so that everything is independent of $\tau$.

To conclude, given any Kähler-Einstein four-metric $g_{\mathrm{KE}}$ with Ricci curvature Ric $= 2\gamma g_{\mathrm{KE}}$, a solution to the ODE system (2.103) leads to a (local) supersymmetric AdS$_4$ solution with internal seven-metric being

$$\begin{aligned} g_7 &= \frac{f_1 f_2}{4\sqrt{1+f_2^2(1+r^2)}} g_{\mathrm{KE}} + \left(\frac{f_2}{4}\right)^2 \left[\mathrm{d}r^2 + \frac{r^2 f_1^2}{1+r^2}(\mathrm{d}\tau + A_{\mathrm{KE}})^2 \right. \\ &\quad \left. + \frac{1+r^2}{1+f_2^2(1+r^2)}\left(\mathrm{d}\psi + \frac{f_1}{1+r^2}(\mathrm{d}\tau + A_{\mathrm{KE}})\right)^2\right] , \end{aligned} \tag{2.105}$$

and flux

$$\begin{aligned} F &= \frac{m^2 \mathrm{e}^{-3\Delta} f_2}{3^3 \cdot 2^7}\left(\gamma m \mathrm{e}^{-3\Delta} f_2 (1+r^2) - 9r^2 f_1\right)(\mathrm{d}\psi - \mathrm{d}\tau)\wedge \frac{\mathrm{d}r}{r}\wedge \mathbb{J}_1 \\ &+ \frac{\gamma m^3 \mathrm{e}^{-6\Delta} f_1 f_2^2}{3^3\cdot 2^7}(\mathrm{d}\tau + A_{\mathrm{KE}})\wedge\left(\frac{\mathrm{d}r}{r}\wedge\mathbb{J}_1 + (\mathrm{d}\psi - \mathrm{d}\tau)\wedge \mathbb{J}_2\right) , \end{aligned} \tag{2.106}$$

where we have written the latter in terms of three functions $f_1, f_2, \mathrm{e}^{\Delta}$ in order to simplify the expression slightly. However, recall that the warp factor is related to $f_2$ via

$$\mathrm{e}^{6\Delta} = \left(\frac{m}{6}\right)^2\left(1 + r^2 + f_2^{-2}\right) . \tag{2.107}$$

Here we have denoted $A_{\mathrm{KE}} \equiv \hat{\mathbb{B}}$, and without loss of generality we have set $\mathbb{B}_\tau = 1$ by rescaling the $\tau$ coordinate. From (2.106) we see explicitly that $\mathcal{L}_{\partial_\tau} F \neq 0$, since the holomorphic two-form on the Kähler-Einstein base satisfies $\mathcal{L}_{\partial_\tau}(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2) = \mathrm{i}\gamma(\mathbb{J}_1 + \mathrm{i}\mathbb{J}_2)$. Therefore, as anticipated at the beginning of this section, $\partial_\tau$ does not generate a symmetry of the full solution. If $\gamma > 0$ then by rescaling $f_1$ we may also without loss of generality set $\gamma = 3$. The local one-form $\gamma A_{\mathrm{KE}}$ is globally a connection on the anti-canonical bundle of the Kähler-Einstein four-space. Notice that we may algebraically eliminate $f_2(r)$ from the first equation in (2.103) to obtain the single second order ODE for $f_1(r)$

$$3rf_1'^2 + f_1(rf_1'' - f_1') = \gamma\sqrt{-2f_1'\left(rf_1 - 2(1+r^2)f_1'\right)} . \tag{2.108}$$

### 2.4.2 The Corrado-Pilch-Warner solution

We begin by noting that the following is an explicit solution to the ODE system (2.103)

$$f_1(r) = \gamma\left(2-\frac{r}{\sqrt{2}}\right) , \qquad f_2(r) = \sqrt{\frac{2}{r(2\sqrt{2}-r)}} . \tag{2.109}$$

Taking the Kähler-Einstein metric to be simply the standard Fubini-Study metric on $\mathbb{CP}^2$, and with $r \in [0, 2\sqrt{2}]$, we claim this is precisely the AdS$_4 \times S^7$ solution described in [27]. In fact the authors of [27] conjectured that one should be able to replace $\mathbb{CP}^2$ by any other Kähler-Einstein metric (with positive Ricci curvature) to obtain another supergravity solution. This was shown in [31] for the special case in which one uses the Kähler-Einstein metrics associated to the $L^{abc}$ Sasaki-Einstein manifolds [57, 58]. We can immediately read off the warp factor

$$\frac{m}{6}\mathrm{e}^{-3\Delta} = \frac{f_2}{\sqrt{1+f_2^2(1+r^2)}} = \frac{1}{1+\frac{r}{\sqrt{2}}} . \tag{2.110}$$

Comparing our d$r^2$ component of the metric (2.105) to the d$\mu^2$ component of the metric in [31], we are led to the identification

$$r = 2\sqrt{2}\sin^2\mu . \tag{2.111}$$

It is then straightforward to see that our metric (2.105) coincides with the metric in [31], and using (2.106) also that the fluxes agree.

### 2.4.3 Deformations of CY$_3 \times \mathbb{C}$ backgrounds

The Corrado-Pilch-Warner solution fits into a more general class of solutions obtained by deforming the theory on $N$ M2-branes at the conical singularity of the Calabi-Yau four-fold CY$_3 \times \mathbb{C}$. In this section we give a unified treatment, in particular recovering the field theory result in [25] for the free energy of such theories using our contact volume formula (2.81).

We begin by taking $g_{\mathrm{KE}}$ to be the (local) Kähler-Einstein metric associated to a Sasaki-Einstein five-manifold. The corresponding Sasaki-Einstein five-metric is

$$g_{\mathrm{SE}_5} = (\mathrm{d}\varphi + A_{\mathrm{KE}})^2 + g_{\mathrm{KE}} , \tag{2.112}$$

which leads to a Calabi-Yau four-fold product metric on CY$_3 \times \mathbb{C}$ given by

$$g_{\mathrm{CY}_4} = \mathrm{d}\rho_1^2 + \rho_1^2\left[(\mathrm{d}\varphi + A_{\mathrm{KE}})^2 + g_{\mathrm{KE}}\right] + \mathrm{d}\rho_0^2 + \rho_0^2\mathrm{d}\varphi_0^2 . \tag{2.113}$$

Here $\rho_0, \rho_1 \in [0,\infty)$ are radial variables, and $\varphi_0$ has period $2\pi$. The corresponding

Sasaki-Einstein seven-metric at unit distance from the conical singularity at $\{\rho_0=\rho_1=0\}$ is

$$g_{SE_7} \;=\; \frac{1}{1-r^2}\mathrm{d}r^2+r^2\Big[(\mathrm{d}\varphi+A_{KE})^2+g_{KE}\Big]+(1-r^2)\mathrm{d}\varphi_0^2\ , \tag{2.114}$$

where $0\leq r\leq 1$. Note that the Killing vector fields $\partial_\varphi$ and $\partial_{\varphi_0}$ vanish at $r=0$ and $r=1$, respectively, and that the Reeb vector field is the sum $\xi=\partial_\varphi+\partial_{\varphi_0}$. The metric (2.114) is singular at $r=0$ (which is an $S^1$ locus parametrized by $\varphi_0$) unless the original Sasaki-Einstein five-manifold is $S^5$ equipped with its standard round metric. This is simply because the Calabi-Yau four-fold is also singular along $r=0$, which is the conical singularity of $CY_3$.

It is no coincidence that the Sasaki-Einstein metric (2.114) resembles our general metric (2.105). The $AdS_4\times SE_7$ background is the infrared limit of $N$ M2-branes at the conical singularity $\{\rho_0=\rho_1=0\}$ of $CY_3\times\mathbb{C}$. The holomorphic function $z_0=\rho_0 e^{i\varphi_0}$ leads to a scalar Kaluza-Klein mode on the Sasaki-Einstein seven-space, which in turn is dual to a gauge-invariant scalar chiral primary operator $\mathcal{O}$ in the dual three-dimensional SCFT. We may then consider deforming the SCFT by adding the operator $\lambda\mathcal{O}^p$. In three dimensions, this is a relevant deformation for $p=2$ and $p=3$, as discussed in [25]. Moreover, such a term can appear in the superpotential of a putative infrared fixed point also only if $p=2$, $p=3$, since otherwise one violates the unitarity bound – the R-charge/scaling dimension of $\mathcal{O}$ would be $\Delta(\mathcal{O})=2/p$, and necessarily we have $\Delta(\mathcal{O})\geq\frac{1}{2}$ for a unitary CFT in three dimensions, with equality only for a free field. The gravity dual to the infrared fixed point of the massive $p=2$ deformation is the Corrado-Pilch-Warner solution of the previous section, while we will find the $p=3$ solution as a numerical solution to the ODEs (2.103) in the next section.

In [25] the authors studied $d=3$, $\mathcal{N}=2$ supersymmetric field theories for $N$ M2-branes on $CY_3\times\mathbb{C}$ backgrounds, in particular computing the free energy using localization and matrix model techniques. This allows one to compute the ratio of UV and IR free energies, where the UV theory is dual to the $AdS_4\times SE_7$ background, while the IR theory is the fixed point of the renormalization group flow induced by the $\lambda\mathcal{O}^p$ deformation. They found the universal formula, independent of the choice of $CY_3$,

$$\frac{\mathcal{F}_{IR}}{\mathcal{F}_{UV}} \;=\; \frac{16(p-1)^{3/2}}{3\sqrt{3}p^2}\ . \tag{2.115}$$

We now show that this field theory result is easily obtained using our contact volume formula (2.81), thus acting as a check of the AdS/CFT duality for this class of theories. The $CY_3\times\mathbb{C}$ Calabi-Yau four-fold has at least a $\mathbb{C}^*\times\mathbb{C}^*$ symmetry,in which the first $\mathbb{C}^*$ acts on the $CY_3$, and under which the $CY_3$ Killing spinors have charge $\frac{1}{2}$, and the second $\mathbb{C}^*$ acts in the obvious way on the copy of $\mathbb{C}$ with coordinate $z_0$. Let us denote the components of the Reeb vector field in this basis as $(\xi_1,\xi_0)$. In terms of the explicit coordinates introduced above, this gives the Reeb vector field as

$$\xi \;=\; \frac{1}{3}\xi_1\partial_\varphi + \xi_0\partial_{\varphi_0}\ . \tag{2.116}$$

For the Calabi-Yau four-fold metric, we have already noted that $\xi_1=3$ and $\xi_0=1$. In general, the Killing spinors have charge 2, as in equation (2.23), precisely when

$$\xi_1+\xi_0 \;=\; 4\ , \tag{2.117}$$

which is also equivalent to the holomorphic $(4,0)$-form $\Omega_{(4,0)}=\Omega_{(3,0)}\wedge \mathrm{d}z_0$ having charge 4. As shown in appendix B of [52], in general the contact volume is a function of the Reeb vector field. In our case the contact volume of $Y_7$ is given by the general formula

$$\mathrm{Vol}(Y_7)[\xi_1,\xi_0] \;=\; \frac{1}{\xi_0}\mathrm{Vol}(Y_5)[\xi_1]\ , \tag{2.118}$$

where $Y_5$ denotes the five-manifold link of CY$_3$. Using $\xi_1=3$ for a Sasaki-Einstein metric, (2.118) implies the relation $\mathrm{Vol}(\mathrm{SE}_7)=\mathrm{Vol}(\mathrm{SE}_5)$ between Sasaki-Einstein volumes. Notice that $\xi_0=1$ gives the expected scaling dimension $\Delta(\mathcal{O})=\frac{1}{2}$ of a free chiral field.[15]

Let us now consider the IR solution corresponding to the deformation by $\lambda\mathcal{O}^p$. The scaling dimension of $\mathcal{O}$ necessarily changes from $\Delta(\mathcal{O})=\frac{1}{2}$ to $\Delta(\mathcal{O})=2/p$. Since the coordinate $z_0$ gives rise to the Kaluza-Klein mode leading to this BPS operator, this means the charge of $z_0$ under the Reeb vector field at the IR fixed point should be $\xi_0=4/p$. From (2.117) we thus have $\xi_1=4(p-1)/p$. We then compute the contact volumes

$$\begin{aligned}\mathrm{Vol}(Y_7^{(p)}) \;&=\; \frac{1}{\xi_0}\mathrm{Vol}(Y_5)[\xi_1] \;=\; \frac{1}{\xi_0}\left(\frac{\xi_1}{3}\right)^{-3}\mathrm{Vol}(Y_5)[3]\\ &=\; \frac{27p^4}{256(p-1)^3}\mathrm{Vol}(\mathrm{SE}_7)\ .\end{aligned} \tag{2.119}$$

Here we have used that the volume of a contact five-manifold is homogeneous degree $-3$ in the Reeb vector field [52], [60]. Taking the square root and using our free energy formula (2.81), we precisely reproduce the field theory result (2.115)![16]

We conclude by recording that the Reeb vector field (2.116) at the IR fixed point is

$$\xi \;=\; \frac{4(p-1)}{3p}\partial_\varphi + \frac{4}{p}\partial_{\varphi_0}\ . \tag{2.120}$$

This will be crucial in the following sections when we consider the appropriate boundary conditions for the ODEs (2.103).

[15] There is a factor of $\frac{1}{2}$ in going from the geometric scaling dimension under the Euler vector to the scaling dimension $\Delta$ in field theory, *cf.* equation (2.31) of [59].

[16] Notice for $p\geq 4$ this is a somewhat formal agreement, since the IR fixed point is not expected to exist due to the unitarity bound, as explained above.

### 2.4.4 The Corrado-Pilch-Warner solution (again)

Before moving on to the gravity dual of the cubic $p=3$ deformation, let us consider again the explicit $p=2$ Corrado-Pilch-Warner solution. The analysis in the previous section implies that the Reeb vector field should be

$$\xi \quad = \quad 4\partial_\psi \quad = \quad \frac{2}{3}\partial_\varphi + 2\partial_{\varphi_0} \ , \tag{2.121}$$

where $\psi$ is the coordinate in (2.105). This fact is very closely related to the appropriate boundary conditions one needs to impose on the ODEs (2.103) in order to obtain a good supergravity solution. For the explicit solution in section 2.4.2, the coordinate $r \in [0, 2\sqrt{2}]$, and by definition $\partial_{\varphi_0}$ is the Killing vector field that vanishes at $r=0$, while $\partial_\varphi$ vanishes at $r=2\sqrt{2}$. Let us see how this works precisely. Without loss of generality we henceforth set

$$\gamma \quad = \quad 3 \ . \tag{2.122}$$

Near to $r=0$, we may use $f_1(0)=2\gamma$, $f_2(r)=2^{-1/4}r^{-1/2}+\mathcal{O}(r^{1/2})$ to compute

$$\left\|A\partial_\psi + B\partial_\tau\right\|^2 |_{r=0} \quad = \quad \frac{1}{16}\left(A+2\gamma B\right)^2 \ . \tag{2.123}$$

This vanishes only if $A=-2\gamma B$, so that the vanishing vector field at $r=0$ is

$$\partial_{\varphi_0} \quad \propto \quad -2\gamma\partial_\psi + \partial_\tau \ . \tag{2.124}$$

To determine the proportionality constant we need to examine the rate of collapse. Introducing $r=4\sqrt{2}R^2$, we have near to $r=0$ that $(\frac{f_2}{4})^2\mathrm{d}r^2=\mathrm{d}R^2[1+\mathcal{O}(R^2)]$. Thus $R$ measures geodesic distance from $R=0$, to leading order, and if $\partial_{\varphi_0}$ is such that $\varphi_0$ has period $2\pi$ and $\partial_{\varphi_0}$ vanishes at $R=0$, then the metric will be smooth here only if $\left\|\partial_{\varphi_0}\right\|=R$. Said another way, to leading order near to $R=0$ the metric must be the standard metric $\mathrm{d}R^2+R^2\mathrm{d}\varphi_0^2$ on $\mathbb{R}^2$ in polar coordinates $(R,\varphi_0)$. We then compute

$$\left\|-2\gamma\partial_\psi + \partial_\tau\right\|^2 \quad = \quad \gamma^2R^2+\mathcal{O}(R^4) \ . \tag{2.125}$$

This fixes

$$\partial_{\varphi_0} \quad = \quad 2\partial_\psi - \frac{1}{\gamma}\partial_\tau \ . \tag{2.126}$$

We may perform a similar analysis near to $r=2\sqrt{2}$. Introducing $2\sqrt{2}-r\equiv 4\sqrt{2}Z^2$, we have $f_1=4\gamma Z^2$, while near to $Z=0$ we have $f_2=2^{-3/2}Z^{-1}+\mathcal{O}(Z)$. Now

$$\left\|A\partial_\psi + B\partial_\tau\right\|^2 \quad = \quad \frac{1}{16\cdot 9}\left[9A^2+\mathcal{O}(Z^2)\right] \ , \tag{2.127}$$

so this vector field vanishes at $r=2\sqrt{2}$ only if $A=0$, leading to

$$\partial_\varphi \quad \propto \quad \partial_\tau \ . \tag{2.128}$$

In particular, the coefficient may be computed from $(\frac{f_2}{4})^2 \mathrm{d}r^2 = \mathrm{d}Z^2[1+\mathcal{O}(Z^2)]$ and

$$\left\|\partial_\tau\right\|^2 \quad = \quad \frac{\gamma^2}{9} Z^2 \ = \ Z^2 \ , \tag{2.129}$$

where we have used $\gamma=3$ in the last step. This is indeed the expected result, since for the canonical scaling of $\gamma=3$ the connection term $\mathrm{d}\tau+\mathbb{A}_{\mathrm{KE}}$ in the metric (2.105) must be the contact one-form $\mathrm{d}\varphi+\mathbb{A}_{\mathrm{KE}}$ for the original Sasaki-Einstein five-manifold (2.112), implying that indeed $\partial_\tau=\partial_\varphi$. The collapsing part of the metric near to $r=2\sqrt{2}$ is then $\mathrm{d}Z^2+Z^2((\mathrm{d}\tau+\mathbb{A}_{\mathrm{KE}})^2+g_{\mathrm{KE}})$. This locally is precisely the CY$_3$ conical metric, giving a smooth collapse at $Z=0$ if and only if the the Kähler-Einstein metric is the standard metric on $\mathbb{CP}^2$. More generally, $r=2\sqrt{2}$ is an $S^1$ locus of CY$_3$ cone singularities.

To summarize, putting (2.126) together with $\partial_\tau=\partial_\varphi$ we have shown

$$2\partial_\psi \quad = \quad \partial_{\varphi_0}+\frac{1}{3}\partial_\varphi \ . \tag{2.130}$$

Recalling that the Reeb vector field is $\xi=4\partial_\psi$, we have thus shown

$$\xi \quad = \quad 4\partial_\psi \ = \ \frac{2}{3}\partial_\varphi+2\partial_{\varphi_0} \ . \tag{2.131}$$

This precisely coincides with (2.121), which was derived in the previous section based only on topological and scaling arguments.

### 2.4.5 Cubic deformations

We may now use precisely the same arguments as the previous section to deduce the appropriate boundary conditions for the ODEs (2.103) in the case of cubic $p=3$ deformations. The Reeb vector field is now

$$\xi \quad = \quad 4\partial_\psi \ = \ \frac{8}{9}\partial_\varphi+\frac{4}{3}\partial_{\phi_0} \ , \tag{2.132}$$

where by definition again $\partial_{\phi_0}$ and $\partial_\varphi$ are the vanishing vector fields, while $\psi$ is the coordinate in our metric (2.105).

Let us begin by considering the behaviour near to $r=0$. Suppose that $f_2(r)=wr^\nu+o(r^\nu)$, with $w$ a non-zero constant. Then the first ODE in (2.103) implies

$$(\log f_1)' \quad \sim \quad -\frac{w^2}{2}r^{1+2\nu} \ , \tag{2.133}$$

which leads to the leading order solution

$$f_1(r) \sim A_0 \exp\left[-\frac{w^2 r^{2(1+\nu)}}{4(1+\nu)}\right] , \tag{2.134}$$

where $A_0$ is a constant. The second ODE in (2.103) is then to leading order

$$\gamma \quad \sim \quad \frac{A_0 w r^\nu(-\nu + w^2 r^{2(1+\nu)}) \exp\left[-\frac{w^2 r^{2(1+\nu)}}{4(1+\nu)}\right]}{\sqrt{1 + w^2 r^{2\nu}(1+r^2)}} . \tag{2.135}$$

For $\nu > 0$ the right hand side tends to zero as $r \to 0$, which is a contradiction. This is also the case for $\nu = 0$. On the other hand, $f_1(r)$ blows up exponentially at $r = 0$ unless $\nu > -1$. Since we do not want the size of the Kähler-Einstein metric to blow up on $Y_7$, a regular solution must hence have $-1 < \nu < 0$. Given this, to leading order the last equation becomes

$$\gamma \quad \sim \quad -A_0 \nu w \left(r^{-2\nu} + w^2\right)^{-1/2} \quad \xrightarrow{r\to 0} \quad -A_0\nu . \tag{2.136}$$

Thus we conclude that $3 = \gamma = -A_0\nu$. Note that $A_0 > 0$, and that the metric (2.105) is then positive definite only if $w > 0$.

As in the previous section, introducing $r = \left(\frac{4(1+\nu)}{w}\right)^{1/(1+\nu)} R^{1/(1+\nu)}$ we compute

$$\left(\frac{f_2}{4}\right)^2 \mathrm{d}r^2 \quad \sim \quad \frac{w^2 r^{2\nu}\mathrm{d}r^2}{16} \quad = \quad \mathrm{d}R^2 , \tag{2.137}$$

We now determine the vanishing vector field at $r = 0$, computing

$$\left\|A\partial_\psi + B\partial_\tau\right\|^2 |_{R=0} \quad = \quad \frac{1}{16}\left(A - \frac{B\gamma}{\nu}\right)^2 , \tag{2.138}$$

where we have eliminated $A_0 = -\gamma/\nu$. Thus the vector field$-\frac{1}{\nu}\partial_\psi - \frac{1}{\gamma}\partial_\tau$ vanishes at $r = 0$. To fix the normalization we need the rate of collapse:

$$\left\|-\frac{1}{\nu}\partial_\psi - \frac{1}{\gamma}\partial_\tau\right\|^2 \quad = \quad \frac{(1+\nu)^2}{\nu^2} R^2 + o(R^2) , \tag{2.139}$$

near to $R = 0$. This fixes

$$\partial_{\varphi_0} \quad = \quad \frac{1}{1+\nu}\partial_\psi + \frac{\nu}{\gamma(1+\nu)}\partial_\tau . \tag{2.140}$$

In fact this is already enough to determine $\nu$. Recall that $\xi = 4\partial_\psi$ is the Reeb vector field, so we can also write

$$\partial_{\varphi_0} \quad = \quad \frac{1}{4(1+\nu)}\xi + \frac{\nu}{\gamma(1+\nu)}\partial_\tau . \tag{2.141}$$

Since the coordinate $z_0$ on $\mathbb{C}$ has charge $2/p$ under $\xi$, we thus conclude that in general

$$1 = \frac{1}{4(1+\nu)} \cdot \frac{4}{p} \ , \tag{2.142}$$

so that

$$\nu = -1 + \frac{1}{p} \ . \tag{2.143}$$

In particular, the Corrado-Pilch-Warner solution has $\nu = -\frac{1}{2}$, while for the cubic deformation we should set $\nu = -\frac{2}{3}$. The boundary condition for $f_2(r)$ near to $r = 0$ is in general $f_2(r) \sim w r^{-1+1/p}$. It is important to note that, with this boundary condition on $f_2(r)$, the metric is completely smooth near to $r = 0$. Although $f_2(r)$ is blowing up, the function $f_1 f_2/\sqrt{1 + f_2^2(1 + r^2)} \sim f_1(0) = -\gamma/\nu$, so that the Kähler-Einstein factor in (2.105) has finite non-zero size. The remaining Killing vector that is not zero also has finite length at $r = 0$, as one sees from (2.138).

We can now similarly analyze the other collapse. This is necessarily at a zero of $f_1(r)$. To see this, note that the Kähler-Einstein part of the metric (2.105) collapses at either a zero of $f_2$, or a zero of $f_1$ (potentially both). Suppose this is at $r = r_0$. If $f_2 \sim \upsilon(r_0 - r)^\eta$ to leading order, with $\eta > 0$, then solving the ODE for $f_1$ leads to the leading order result

$$f_1(r) \sim A_1 \exp\left[\frac{\upsilon^2 r_0 (r_0 - r)^{1+2\eta}}{2(1 + 2\eta)}\right] \ . \tag{2.144}$$

Thus $f_1(r_0) = A_1$ is in fact non-zero. The second ODE in (2.103) is then consistent near to $r = r_0$ only if the exponent $\eta = 1$, which means that $f_2(r) \sim \upsilon(r_0 - r)$ is a simple zero. However, from the metric (2.105) we see that in fact then the entire metric collapses at $r = r_0$, which does not give the correct topology. So we can rule out $f_2(r)$ having a zero at $r = r_0$.

Thus $f_1(r_0) = 0$. Let us suppose that to leading order

$$f_1(r) \sim q(r_0 - r)^\lambda \ , \tag{2.145}$$

with $\lambda > 0$. Then from the first ODE in (2.103) we obtain

$$f_2(r) \sim \sqrt{\frac{2\lambda}{r_0(r_0 - r)}} \ . \tag{2.146}$$

Notice that for the Corrado-Pilch-Warner solution we have $\lambda_{\mathrm{CPW}} = 1$, and this leading order solution for $f_2(r)$ near to $r = r_0$ is in fact the *exact* solution. For our cubic $p = 3$ solution $f_2(r)$ must instead interpolate between $r^{-2/3}$ behavior near to $r = 0$ and $(r_0 - r)^{-1/2}$ behavior near to $r = r_0$. The second ODE again fixes the exponent $\lambda = 1$ for consistency near to $r=r_0$, and we conclude that

$$f_1(r) \sim q(r_0-r) \,, \tag{2.147}$$

$$f_2(r) \sim \sqrt{\frac{2}{r_0(r_0-r)}} \,, \tag{2.148}$$

near to $r=r_0$. Moreover, the second ODE then fixes

$$\gamma = \frac{3qr_0}{2\sqrt{1+r_0^2}} \,. \tag{2.149}$$

Finally, we turn to looking at the vanishing vector field. Writing $r_0-r\equiv 2r_0W^2$, we find that $(\frac{f_2}{4})^2 \mathrm{d}r^2 \sim \mathrm{d}W^2$. Then

$$\left\|A\partial_\psi + B\partial_\tau\right\|^2 = \frac{1}{16}A^2 + \mathcal{O}(W^2) \,, \tag{2.150}$$

so that the vanishing vector field at the root $r=r_0$ is again proportional to $\partial_\tau$. We find more precisely that, quite remarkably,

$$\left\|\partial_\tau\right\|^2 = \left(\frac{\gamma}{3}\right)^2 W^2 + o(W^2) \,, \tag{2.151}$$

where we have substituted for $q$ using (2.149). This is exactly the same behavior as for the Corrado-Pilch-Warner solution near to this root. Since this collapsing vector field is by definition $\partial_\varphi$, we again conclude that

$$\partial_\tau = \partial_\varphi \,. \tag{2.152}$$

Again, this had to be the case for global reasons associated to the form of the connection one-form appearing in the metric. Again one finds that $r=r_0$ is an $S^1$ family of CY$_3$ cone singularities, with the analysis being identical to that for the Corrado-Pilch-Warner solution in the previous section.

This completes our analysis of the regularity conditions. Setting $\gamma=3$, we have shown that the Reeb vector field is

$$\xi = -\frac{4\nu}{3}\partial_\varphi + 4(1+\nu)\partial_{\phi_0} \,. \tag{2.153}$$

Using the fact that $\nu=-1+\frac{1}{p}$, this precisely agrees with our topological analysis in subsection 2.4.3, and in particular the formula (2.120).

### 2.4.6 Summary and numerics

We may summarize the results of the previous sections as follows.

*The gravity dual to the infrared fixed point of a deformation of a $CY_3 \times \mathbb{C}$ background by the operator $\lambda\mathcal{O}^p$ may be obtained by solving the coupled set of ODEs for $f_2(r)$, $f_1(r)$:*

$$\begin{aligned} f_1' &= -\tfrac{1}{2} r f_1 f_2^2 \ , \\ f_2' &= -3r^{-1}\sqrt{1+f_2^2(1+r^2)} + r f_1 f_2^3 \ . \end{aligned} \tag{2.154}$$

*The boundary conditions are that near to $r=0$ we have $f_2(r) \sim w r^{-1+1/p}$, with $w>0$ a constant. Using the second ODE above this implies that $f_1(0) = 3p/(p-1)$. Then near to $r=r_0$, for some $r_0>0$, we must impose that $f_1(r) \sim q(r_0 - r)$, where the ODEs imply that $f_2(r) \sim \sqrt{2/r_0(r_0-r)}$ and $q = 2\sqrt{1+r_0^2}/r_0$. With these boundary conditions we obtain a smooth supergravity solution, up to the expected $S^1$ locus of $CY_3$ singularities along $r=r_0$. When the $CY_3$ is simply flat $\mathbb{C}^3$, in particular we obtain a completely smooth $\mathcal{N}=2$ supergravity solution with the topology $AdS_4 \times S^7$.*

The Corrado-Pilch-Warner solution precisely solves this problem for $p=2$, and physical arguments imply there should also be a solution for $p=3$. We have not been able to find this solution analytically, but it is straightforward to solve the ODEs numerically with the above boundary conditions.

We first change variable to $r=R^3$, and then solve the second order ODE (2.108) in a Taylor expansion in $R$, around $R=0$, up to some large order. Using the constraint that $f_1(0) = 3p/(p-1) = 9/2$ we find

$$f_1(R) \ = \ \frac{9}{2} - cR^2 - \frac{c^2}{9}R^4 + \frac{2187-128c^3}{3888}R^6 + \frac{19683c + 1264c^3}{104976}R^8 + \mathcal{O}(R^{10}) \ , \tag{2.155}$$

where $c$ is an arbitrary integration constant. This then implies

$$f_2(R) \ = \ \frac{2}{3}\sqrt{\frac{2}{3}}c^{1/2}R^{-2} + \frac{4}{27}\sqrt{\frac{2}{3}}c^{3/2} - \frac{(2187-224c^3)}{1944\sqrt{6}}c^{-1/2}R^2 + \mathcal{O}(R^4) \ . \tag{2.156}$$

Thus $f_2(r)$ has the correct behavior $f_2(r) \sim w r^{-2/3}$, where we identify the constant $w = \sqrt{8c/27}$.

We then have a numerical shooting problem: for each choice of integration constant $c$, we solve the second order ODE (2.108) (or equivalently the coupled first order system), with initial Taylor expansion (2.155). We simply require that $f_1(r_0)=0$ for some $r_0>0$. From the analysis in the previous section, the ODEs themselves imply that a zero of $f_1(r)$ is automatically a simple zero.

We find that there exists a point $r_0>0$ with $f_1(r_0)=0$ for the choice

$$c \ \simeq \ 2.4998 \ . \tag{2.157}$$

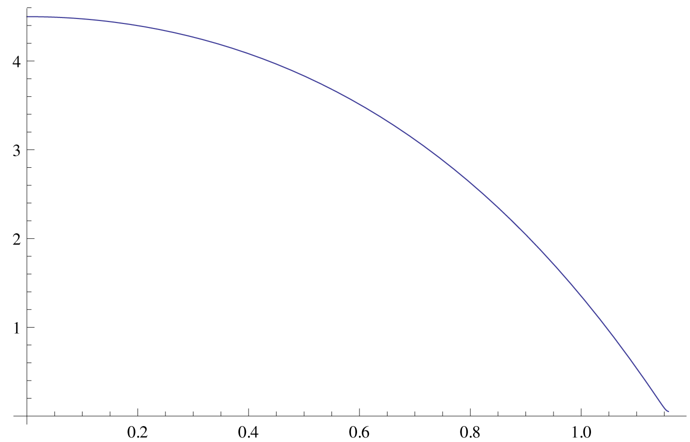


Figure 2.1: Numerical plot of the function $f_1(R)$ with integration constant $c \simeq 2.4998$. Note that $f_1(0) = 9/2$ and $f_1(R)$ decreases monotonically to zero at $R = R_0$, where $R_0 \simeq 1.16$.

The resulting plot of the function $f_1(R)$, with $R = r^3$, is shown in Figure 2.1. Smaller values of $c$ lead to $f_1(R)$ remaining positive, while for $c > 2.4998$ we find the numerics becomes highly unstable. Indeed, the numerics is slightly unstable near the zero of $f_1$ for $c = 2.4998$. As a cross check that we really do have a zero, we note that at a zero of $f_1(R)$ we necessarily have

$$f_1'(R_0) \quad = \quad -\frac{6\sqrt{1+R_0^6}}{R_0} \ . \tag{2.158}$$

In Figure 2.2 we numerically plot the function $f_1'(R) + \frac{6\sqrt{1+R^6}}{R}$, which should tend to zero at $R = R_0$.

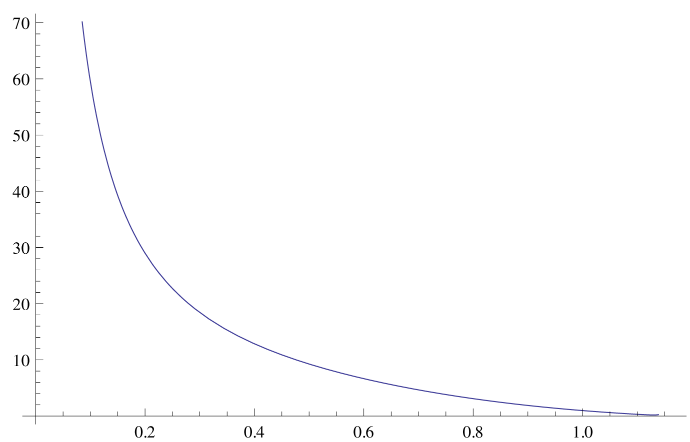


Figure 2.2: Numerical plot (with integration constant $c \simeq 2.4998$) of the function $f_1'(R) + \frac{6\sqrt{1+R^6}}{R}$, which should tend to zero at $R = R_0 \simeq 1.16$.

Of course, it is quite tantalizing that the numerical value of $c$ is so close to 5/2, perhaps suggesting the possibility of an analytic solution, or at least an analytic explanation of $c = 5/2$. We leave this question open.

## 2.5 Special cases

### 2.5.1 The Sasaki-Einstein case

In this subsection we study the case in which the three one-forms $K, \mathrm{Re}\, S^*L, \mathrm{Im}\, S^*L$ are linearly dependent. When they are linearly independent we have an $SU(2)$ structure, and in an open set we can then introduce corresponding coordinates, as described in section 2.2.5. Since these one-forms are derived from spinor bilinears, linear dependence implies we have an $SU(3)$ structure. Focusing on the $m \neq 0$ case for clarity, we will prove that the only solutions for which we have a global $SU(3)$ structure are Sasaki-Einstein.

In order to proceed, we impose the linear relation

$$a K + b\, \mathrm{Re}\, S^*L + c\, \mathrm{Im}\, S^*L \;\; = \;\; 0 \;, \tag{2.159}$$

with $a$, $b$, $c$ not all zero. Making use of the Fierz identity in (A.6) it is straightforward to compute the dot products of each of $K, \mathrm{Re}\, S^*L, \mathrm{Im}\, S^*L$ into this equation. An analysis of the resulting three equations then implies that at least one of $|S| = 0$ or $\|\xi\| = 1$ must hold. In particular, if $|S| = 0$ then necessarily $a = 0$, while if $\|\xi\| = 1$ then $a = c(\frac{m^2}{36}\mathrm{e}^{-6\Delta} - 1)$. The following analysis then treats these cases in turn.

If $|S| = 0$ then of course also $S = 0$. The bilinear equation (2.24) then implies that $L = 0$ and hence in particular that the one-form $\bar{\chi}_1\gamma_{(1)}\chi_1 = 0$. This says that $\chi_1$ defines a $G_2$ structure, rather than an $SU(3)$ structure, and hence that $\chi_1$ satisfies a reality (Majorana) condition $\chi_1 = \mu\chi_1^c$. The scalar bilinears determine that $\mu = -\frac{\mathrm{i}6}{m}\mathrm{e}^{3\Delta}$, and since $|\mu|^2 = 1$ we conclude that $\frac{m}{6}\mathrm{e}^{-3\Delta} = 1$ and the warp factor is constant $\mathrm{e}^{3\Delta} = m/6$. Finally, the bilinear equation (2.28) and its $\chi_-$ analogue imply

$$\mathrm{e}^{3\Delta} \star F \;\; = \;\; \mathrm{d}\left(\mathrm{i}\, \mathrm{e}^{6\Delta}\bar{\chi}_1\gamma_{(2)}\chi_1\right) - 6\mathrm{e}^{6\Delta}\mathrm{Im}\left[\bar{\chi}_1^c\gamma_{(3)}\chi_1\right] \;, \tag{2.160}$$

which in turn immediately implies that $F = 0$. This is because the Majorana condition $\chi_1 = -\mathrm{i}\chi_1^c$ implies that the two-form bilinear $\bar{\chi}_1\gamma_{(2)}\chi_1 = 0$ (there are no $G_2$-invariant two-forms), while the three-form bilinear $\bar{\chi}_1^c\gamma_{(3)}\chi_1$ is real (corresponding to the unique $G_2$-invariant three-form). We conclude that the warp factor is constant and $F = 0$, so that the Killing spinor equation for $\chi_1$ (2.8) leads to weak $G_2$ holonomy and hence an Einstein metric. The second Killing spinor $\chi_2$ (for which the analysis is essentially the same) then of course leads to a Sasaki-Einstein manifold.

Alternatively, if $\|\xi\| = 1$ then we immediately have $\mathrm{Re}\, S^*L = 0$ by computing the square length of the latter using (A.6). But since also $a = -c|S|^2$ follows from linear dependence, we also have the additional relation $\mathrm{Im}\, S^*L = |S|^2 K$ from (2.159). There is thus only one linearly independent vector, as one expects since we must have an $SU(3)$ structure. Using the exterior derivatives of the one-form bilinears one can then show that where $S$ is non-zero we have that $K$ is closed, $\mathrm{d}K = 0$ (recall that $K$ is Killing in any case, so this implies that $K$ is parallel). By contracting $K$ into the bilinear equation for $\mathrm{d}K$ and making use of a Fierz identity one then proves that $\mathrm{d}\Delta = 0$. Given that $\|\xi\|^2 = |S|^2 + \frac{m^2}{36}\mathrm{e}^{-6\Delta} = 1$ by assumption, this immediately implies that $S$ is constant, and hence that $L = 0$. But then all vectors are identically zero, and we have a contradiction. Thus it must be that $S = 0$ and we hence reduce to the previous case, which implies that $Y_7$ is Sasaki-Einstein with $F = 0$ and $\Delta$ constant.

### 2.5.2 The case $m = 0$, $\mathrm{Im}[\bar{\chi}_1\chi_2] \neq 0$

In section 2.2.2 we noted that when $m = 0$ we can no longer conclude that equation (2.15) holds. In this appendix we study the case $m = 0$ but $\mathrm{Im}[\bar{\chi}_1\chi_2]$ not being identically zero, in particular showing that there are no regular solutions in this class. Note this is different from the class of $m = 0$ geometries discussed in section 2.2.7, and cannot be obtained by taking the $m \to 0$ limit of the general $m \neq 0$ equations in the main text.

We begin by defining

$$h \equiv \mathrm{Im}[\bar{\chi}_1\chi_2] , \tag{2.161}$$

which is a function on $Y_7$. Equation (2.18) now becomes

$$\mathrm{Im}\, K = \frac{1}{2}\mathrm{d}h , \tag{2.162}$$

while the imaginary part of equation (2.19) reads

$$\nabla_{(m}(\mathrm{Im}\, K)_{n)} = -2hg_{7\,mn} . \tag{2.163}$$

Combining the last two equations gives

$$\nabla_m\nabla_n h = -t^2 hg_{7\,mn} , \tag{2.164}$$

where $t = 2$. Notice that $\mathrm{Im}\, K$ is a particular type of gradient conformal Killing vector. Equation (2.164) was studied by Obata in [61]. In particular, he proved that if a complete Riemannian manifold of dimension $d \geq 2$ admits a non-constant function $h$ satisfying (2.164), where $t$ is (without loss of generality) a positive constant, then it is necessarily isometric to a round sphere of radius $1/t$. Thus we immediately conclude that if $h$ is not identically zero, $Y_7$ is isometric to the round $S^7$ with radius 1/2.

Now as in section 2.2.7, the Bianchi identity and equation of motion for $F$ imply that $F$ is harmonic on the conformally rescaled manifold $(Y_7, \tilde{g}_7)$, where $\tilde{g}_7 = \mathrm{e}^{-6\Delta}g_7$. But in the case at hand, $Y_7 = S^7$ and the Hodge theorem implies there are no harmonic four-forms since $H^4(S^7; \mathbb{R}) = 0$. Thus for a non-singular solution in fact $F = 0$, and hence the M-theory four-form $G = 0$. The equation of motion (2.2) then implies that the eleven-dimensional spacetime must be Ricci-flat, but this is a contradiction.

# 3 The supersymmetric NUTs and bolts of holography

## 3.1 Preview

In this chapter we construct gravity duals to field theories on a biaxially squashed three-sphere considered in [38] and [39]. More generally, we will perform an exhaustive study of supersymmetric asymptotically locally AdS$_4$ solutions whose conformal boundary is given by a biaxially squashed *Lens space* $S^3/\mathbb{Z}_p$. We will first work within (Euclidean) minimal gauged supergravity in four dimensions, determining the general local form of the supersymmetric solutions with $SU(2) \times U(1)$ symmetry, and then we will discuss in detail the global properties of these solutions, both in four dimensions and in eleven-dimensional supergravity. Despite the high degree of symmetry of the problem, we uncover a surprisingly intricate web of supersymmetric solutions. One of our main findings is that generically a given conformal boundary can be "filled" with more than one supersymmetric solution, with different topology. More specifically, we will show that for a given choice of conformal class of metric and gauge field there exist supersymmetric solutions with the topology of $\mathbb{R}^4$ (or $\mathbb{Z}_p$ orbifolds of this) - the *NUTs* - and different supersymmetric solutions with the topology of $\mathcal{M}_p \equiv$ total space of $\mathcal{O}(-p) \to S^2$ - the *bolts*.[1] The discussion of these Taub-Bolt-AdS solutions is subtle: they typically exist only in certain ranges of the squashing parameter, depending on $p$ and the amount of supersymmetry preserved, and moreover typically they have globally different boundary conditions to the corresponding $\mathbb{Z}_p$ quotient of a Taub-NUT-AdS solution (related to the addition of a flat Wilson line at infinity for the gauge field). Appealing to a conjecture [62] that the (conformal) isometry group of the conformal boundary extends to the isometry of the bulk,[2] we will have found *all* possible supersymmetric fillings of a given boundary, at least in the context of four-dimensional minimal gauged supergravity.

The results we find have interesting implications for the AdS/CFT correspondence. Recall that when there exist inequivalent fillings of a fixed boundary one should sum over all the contributions in the saddle point approximation to the path integral. Equivalently, the partition function of the dual field theory (in the large $N$ limit) is given by the

[1] In particular, $H_2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$ and there is hence a non-trivial two-cycle, which is referred to as a "bolt".
[2] See also Appendix B of [63].

*sum* of the exponential of minus the supergravity action, evaluated on each solution with a fixed boundary. If different solutions dominate the path integral (have smallest free energy) in different regimes of the parameters, then passing from one solution to another is interpreted as a phase transition between vacua of the theory. In the example of the Hawking-Page phase transition [64], discussed in [65], the two gravity solutions with the same boundary are thermal $\mathrm{AdS}_4$ and the Schwarzchild-$\mathrm{AdS}_4$ solution, and the parameter being dialed is the temperature of the black hole (or equivalently of the dual field theory). The more sophisticated examples discussed in [66, 67] share a number of similarities with the results presented here, but there are some crucial differences. The latter references studied Taub-NUT-AdS and Taub-Bolt-AdS solutions, whose conformal boundary metric is precisely the biaxially squashed three-sphere. However, these are all non-supersymmetric Einstein solutions, and do not possess any gauge field.[3] On the other hand, our solutions will all have a non-trivial gauge field turned on, which is necessary in order to preserve supersymmetry. We will therefore refrain from interpreting the squashing parameter as the inverse temperature. Whether or not one should sum over our Taub-Bolt-AdS solutions, in the saddle point approximation to quantum gravity, depends on whether they are interpreted as different vacua of the same theory, or rather as vacua of (subtly) different field theories. This in turn depends on the uplifting of the solutions to M-theory, discussed briefly in the next paragraph, but we shall argue that,at least in some cases, the Taub-Bolt-AdS solutions have (subtly) different boundary conditions to the Taub-NUT-AdS solutions.

An interesting aspect of the supersymmetric Taub-Bolt-AdS solutions (with topology $\mathcal{M}_p$) is that these can be uplifted to solutions $\mathcal{M}_p \widetilde{\times} Y_7$ of M-theory only for particular internal Sasaki-Einstein manifolds $Y_7$. Indeed, the key issue here is that $Y_7$ is necessarily fibred over $\mathcal{M}_p$, which we have denoted with the tilde. As we shall explain, for all these solutions the free energy of the field theory has not yet been studied in the literature, and therefore we cannot compare our gravity results with an existing field theory calculation. However, for both classes of solutions of Taub-NUT-AdS type (1/2 BPS and 1/4 BPS), where the dual field theories are placed on squashed three-spheres, we obtain a precise matching between our gravity results and the results from localization in field theory.

## 3.2 $SU(2) \times U(1)$-invariant solutions of gauged supergravity

We begin by presenting all Euclidean supersymmetric solutions of $d = 4$, $\mathcal{N} = 2$ gauged supergravity with $SU(2) \times U(1)$ symmetry. The action for the bosonic sector of this theory

[3] As we shall discuss, the Taub-NUT-AdS metric has self-dual Weyl tensor, and hence it can be made supersymmmetric by adding particular instanton fields [68]. The Taub-Bolt-AdS metric in [66, 67] is not self-dual, and cannot be made supersymmetric by adding any instanton.

[69] reads

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\left(R+6\ell^{-2}-F^2\right) . \tag{3.1}$$

Here $R$ denotes the Ricci scalar of the metric $g_{\mu\nu}$ and we have defined $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$. $G_4$ is the four-dimensional Newton constant and $\ell$ is a parameter with dimensions of length, related to the cosmological constant via $\Lambda = -3\ell^{-2}$. The graviphoton is an Abelian gauge field $A$ with field strength $F = \mathrm{d}A$.

The equations of motion derived from (3.1) read

$$\begin{aligned} R_{\mu\nu} &= -3\ell^{-2}g_{\mu\nu}+2\left(F_\mu{}^\rho F_{\nu\rho}-\frac{1}{4}F^2 g_{\mu\nu}\right) , \\ \mathrm{d}*F &= 0 . \end{aligned} \tag{3.2}$$

In Euclidean signature the gauge field may in principle be complex, although for the solutions in this paper the field strength $F$ will in fact be either real or purely imaginary.[4]

A solution is supersymmetric if there is a non-trivial Dirac spinor $\epsilon$ satisfying the Killing spinor equation

$$\left(\nabla_\mu - \mathrm{i}\ell^{-1}A_\mu+\frac{1}{2}\ell^{-1}\Gamma_\mu+\frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = 0 . \tag{3.3}$$

This takes the same form as in Lorentzian signature, except that here $\Gamma_\mu$, $\mu = 1,2,3,4$, generate the Clifford algebra Cliff(4, 0), so $\{\Gamma_\mu,\Gamma_\nu\} = 2g_{\mu\nu}$. It was shown in [70, 71] that any such solution uplifts (locally) to a supersymmetric solution of eleven-dimensional supergravity. As we will see, global aspects of this uplift can be subtle, and we will postpone a detailed discussion of these issues until section 3.6. In the remainder of this section all computations will be *local*. In what follows we set $\ell = 1$; factors of $\ell$ may be restored by dimensional analysis.

### 3.2.1 General solution to the Einstein equations

Our aim is to find, in explicit form, *all* asymptotically locally AdS$_4$ solutions in Euclidean signature with boundary a biaxially squashed Lens space. Recall that the round metric on $S^3$ has $SU(2)_l\times SU(2)_r$ isometry. A biaxially squashed Lens space is described by an $SU(2)_l\times U(1)_r$-invariant metric on $S^3/\mathbb{Z}_p$, where $\mathbb{Z}_p\subset SU(2)_r$. Given a (conformal) Killing vector field on a compact three-manifold $\mathcal{M}^{(3)}$, a theorem of Anderson [62] shows that this extends to a Killing vector for any asymptotically locally AdS$_4$ Einstein metric on $\mathcal{M}^{(4)}$ with conformal boundary $\mathcal{M}^{(3)} = \partial\mathcal{M}^{(4)}$, provided $\pi_1(\mathcal{M}^{(4)},\mathcal{M}^{(3)}) = 0$. In particular, this result applies directly to the class of *self-dual* solutions that we will discuss momentarily. Anderson also conjectures that this result extends to more general asymptotically locally

[4] In principle the metric may also be complex, although we will not consider that possibility here.

$AdS_4$ solutions to the Einstein-Maxwell equations. Assuming this conjecture holds, we may hence restrict our search to $SU(2) \times U(1)$-invariant solutions.[5]

The general ansatz for the metric and gauge field takes the form

$$\begin{aligned} \mathrm{d}s_4^2 &= \alpha^2(r)\mathrm{d}r^2 + \beta^2(r)(\sigma_1^2 + \sigma_2^2) + \gamma^2(r)\sigma_3^2 , \\ A &= h(r)\sigma_3 , \end{aligned} \tag{3.4}$$

where $\sigma_1, \sigma_2, \sigma_3$ are $SU(2)$ left-invariant one-forms, which may be written in terms of Euler angular variables as

$$\sigma_1 + \mathrm{i}\sigma_2 = \mathrm{e}^{-\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi) , \quad \sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi . \tag{3.5}$$

Note that in the case $h(r) \equiv 0$, when the metric is necessarily Einstein, the general form of the solutions was obtained by Page-Pope [73]. We are not aware of any study of the equations in the most general Einstein-Maxwell case. In appendix C we show that the general solution to (3.2) with the ansatz (3.4) is given by

$$\begin{aligned} \mathrm{d}s_4^2 &= \frac{r^2 - s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2\Omega(r)}{r^2 - s^2}\sigma_3^2 , \\ A &= \left( P\frac{r^2 + s^2}{r^2 - s^2} - Q\frac{2rs}{r^2 - s^2} \right) \sigma_3 , \end{aligned} \tag{3.6}$$

where

$$\Omega(r) = (r^2 - s^2)^2 + (1 - 4s^2)(r^2 + s^2) - 2Mr + P^2 - Q^2 . \tag{3.7}$$

Here $s$, $M$, $P$ and $Q$ are integration constants. This coincides with an analytic continuation of the Reissner-Nordström-Taub-NUT-AdS (RN-TN-AdS) solutions originally found in [74] and [75], and reduces to the Page-Pope metrics for $P^2 - Q^2 = 0$. The supersymmetry properties of the RN-TN-AdS solutions were studied in [76] and [77].

It is a simple matter to check that the metric (3.6) is asymptotically locally $AdS_4$ as $|r| \to \infty$. At large $|r|$ the metric is to leading order

$$\mathrm{d}s_4^2 \approx \frac{\mathrm{d}r^2}{r^2} + r^2 \left( \sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2 \right) , \tag{3.8}$$

so that the conformal boundary at $r = \pm\infty$ is (locally) a biaxially squashed $S^3$.

[5] This result should be contrasted with the corresponding situation for asymptotically locally Euclidean metrics, where Killing vector fields on the boundary do not necessarily extend inside. The canonical examples are the Gibbons-Hawking multi-centre solutions [72].

### 3.2.2 BPS equations

The requirement of supersymmetry imposes constraints on the four parameters $s, M, P$ and $Q$. In appendix D.1 we show that the integrability condition of (2.10) implies

$$D \;=\; 0\;, \qquad B_+ B_- \;=\; 0\;, \tag{3.9}$$

where

$$\begin{aligned} D &\equiv 2\Big[MP - sQ(1-4s^2)\Big]\;, \\ B_\pm &\equiv (M \pm sQ)^2 - s^2(1 \pm P - 4s^2)^2 - (1 \pm 2P - 5s^2)(P^2 - Q^2)\;. \end{aligned} \tag{3.10}$$

These are *necessary* but not sufficient conditions for supersymmetry. One can show that solutions to the algebraic equations (3.10) fall into three classes:

$$\begin{aligned} &\text{Class I}: && M = \pm 2sQ\;, && P = \mp\frac{1}{2}(4s^2-1)\;, \\ &\text{Class II}: && M = \pm Q\sqrt{4s^2-1}\;, && P = \mp s\sqrt{4s^2-1}\;, \\ &\text{Class III}: && M = \mp s(4s^2-1)\;, && P = \pm Q\;. \end{aligned} \tag{3.11}$$

As we will show in the next section by explicitly solving the Killing spinor equation (2.10), Class I corresponds to 1/4 BPS solutions while Class II corresponds to 1/2 BPS solutions. Class III are Einstein but in general *not* supersymmetric, although both Classes II and III satisfy $D = B_+ = B_- = 0$. The upper and lower signs in (3.11) in fact lead to the same (local) solutions for the metric and gauge field: in Class II the upper and lower signs are exchanged by sending $\{r \to -r, s \to -s\}$, while for Class I the upper and lower signs are exchanged by sending $\{r \to -r, \psi \to -\psi, \varphi \to -\varphi\}$. Thus, after a change of variable, the solutions for the metric and gauge field are in fact identical. Without loss of generality we will thus focus on the following two cases:

$$1/4\ \text{BPS}: \quad M = 2sQ\;, \qquad P = -\frac{1}{2}(4s^2-1)\;, \tag{3.12}$$

$$1/2\ \text{BPS}: \quad M = Q\sqrt{4s^2-1}\;, \qquad P = -s\sqrt{4s^2-1}\;. \tag{3.13}$$

### 3.2.3 Killing spinors

In this section we solve the Killing spinor equation (3.3). We will do so separately for the two classes of BPS constraints (3.12), (3.13). In this section we will only derive the form of the Killing spinors in a convenient local orthonormal frame; *global* aspects of these spinors will be addressed later in the paper, and in particular in appendix E. The Einstein metrics in Class III will be discussed further in section 3.3.

We work in the local orthonormal frame

$$
\begin{aligned}
e^1 &= \sqrt{r^2-s^2}\,\sigma_1 \ , \qquad & e^2 &= \sqrt{r^2-s^2}\,\sigma_2 \ , \\
e^3 &= 2s\sqrt{\frac{\Omega(r)}{r^2-s^2}}\sigma_3 \ , \qquad & e^4 &= \sqrt{\frac{r^2-s^2}{\Omega(r)}}\mathrm{d}r \ ,
\end{aligned}
\tag{3.14}
$$

and write $\Omega(r)$ as

$$
\Omega(r) \ = \ (r-r_1)(r-r_2)(r-r_3)(r-r_4) \ . \tag{3.15}
$$

We take the following basis of four-dimensional gamma matrices:

$$
\Gamma_\alpha \ = \ \begin{pmatrix} 0 & \tau_\alpha \\ \tau_\alpha & 0 \end{pmatrix} \ , \qquad \Gamma_4 \ = \ \begin{pmatrix} 0 & \mathrm{i}\mathbb{I}_2 \\ -\mathrm{i}\mathbb{I}_2 & 0 \end{pmatrix} \ , \tag{3.16}
$$

where $\tau_\alpha$, $\alpha = 1, 2, 3$ are the Pauli matrices. Accordingly,

$$
\Gamma_5 \ \equiv \ \Gamma_1\Gamma_2\Gamma_3\Gamma_4 \ = \ \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \ . \tag{3.17}
$$

We decompose the Dirac spinor $\epsilon$ into positive and negative chirality parts as

$$
\epsilon \ = \ \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} \ , \tag{3.18}
$$

and further denote the components of $\epsilon_\pm$ as

$$
\epsilon_\pm \ = \ \begin{pmatrix} \epsilon_\pm^{(+)} \\ \epsilon_\pm^{(-)} \end{pmatrix} \ . \tag{3.19}
$$

**1/2 BPS solutions**

In this section we solve the Killing spinor equation (3.3) for the second class of BPS constraints (3.13). We first obtain an algebraic relation between $\epsilon_+$ and $\epsilon_-$ by using the integrability condition (D.1). In particular, by decomposing (D.1) into chiral parts using the (3.16) basis of gamma matrices we derive

$$
\begin{aligned}
\epsilon_-^{(+)} &= \mathrm{i}\sqrt{\frac{r-s}{r+s}}\sqrt{\frac{(r-r_1)(r-r_2)}{(r-r_3)(r-r_4)}}\epsilon_+^{(+)} \ , \\
\epsilon_-^{(-)} &= \mathrm{i}\sqrt{\frac{r-s}{r+s}}\sqrt{\frac{(r-r_3)(r-r_4)}{(r-r_1)(r-r_2)}}\epsilon_+^{(-)} \ .
\end{aligned}
\tag{3.20}
$$

Here we have identified the roots of $\Omega(r)$ in (3.15) as

$$\left\{ \begin{array}{c} r_4 \\ r_3 \end{array} \right\} = \frac{1}{2}\left[-\sqrt{4s^2-1} \pm \sqrt{8s^2-4Q-1}\right] ,$$
$$\left\{ \begin{array}{c} r_2 \\ r_1 \end{array} \right\} = \frac{1}{2}\left[\sqrt{4s^2-1} \pm \sqrt{8s^2+4Q-1}\right] . \qquad (3.21)$$

We continue by looking at the $\mu = r$ component of the Killing spinor equation. Decomposing this into chiral parts we obtain

$$\partial_r \epsilon_+ = -\frac{\mathrm{i}}{2}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\,\epsilon_- - \mathrm{i}\sqrt{\frac{r^2-s^2}{\Omega(r)}} \cdot \frac{s\sqrt{4s^2-1}+Q}{2(r-s)^2}\,\tau_3\,\epsilon_- ,$$
$$\partial_r \epsilon_- = +\frac{\mathrm{i}}{2}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\,\epsilon_+ + \mathrm{i}\sqrt{\frac{r^2-s^2}{\Omega(r)}} \cdot \frac{s\sqrt{4s^2-1}-Q}{2(r+s)^2}\,\tau_3\,\epsilon_+ . \qquad (3.22)$$

Using the relations (3.20) it is straightforward to solve the above first order ODEs. The general solution is

$$\epsilon_+ = \left( \begin{array}{c} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\chi^{(+)} \\ \sqrt{\frac{(r-r_1)(r-r_2)}{r-s}}\chi^{(-)} \end{array} \right) , \quad \epsilon_- = \mathrm{i}\left( \begin{array}{c} \sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\chi^{(+)} \\ \sqrt{\frac{(r-r_3)(r-r_4)}{r+s}}\chi^{(-)} \end{array} \right) , \qquad (3.23)$$

where the components $\chi^{(\pm)}$ depend only on the angular coordinates. We may then form the $r$-independent two-component spinor

$$\chi \equiv \left( \begin{array}{c} \chi^{(+)} \\ \chi^{(-)} \end{array} \right) , \qquad (3.24)$$

The remaining components of the Killing spinor equation (2.10) then reduce to the following Killing spinor equation for $\chi$:

$$\left(\nabla^{(3)}_\alpha - \mathrm{i}A^{(3)}_\alpha - \frac{\mathrm{i}s}{2}\gamma_\alpha - \frac{\mathrm{i}}{2}\sqrt{4s^2-1}\gamma_\alpha\gamma_3\right)\chi = 0 . \qquad (3.25)$$

Here $\nabla^{(3)}$ denotes the spin connection for the three-metric

$$\mathrm{d}s_3^2 = \sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2 , \qquad (3.26)$$

with $\gamma_\alpha = \tau_\alpha$, $\alpha = 1, 2, 3$ generating the corresponding Cliff$(3, 0)$ algebra in an orthonormal frame, and

$$A^{(3)} = \lim_{r\to\infty} A = P\sigma_3 = -s\sqrt{4s^2-1}\sigma_3 . \qquad (3.27)$$

The three-metric (3.26) and gauge field (3.27) are in fact the conformal boundary of (3.6) at $r = \infty$. It is important to stress here that, in general, the expression (3.27) is valid only *locally*, that is in a coordinate patch. The precise global form of the gauge field, and how this interacts with the spin structure, will be discussed later in the paper, and in particular in appendix E.

The general solution to (3.25) in the orthonormal frame

$$\tilde{e}^1 = \sigma^1 , \qquad \tilde{e}^2 = \sigma^2 , \qquad \tilde{e}^3 = 2s\sigma^3 \tag{3.28}$$

induced from the $r \to \infty$ limit of the frame (3.14) ($\tilde{e}^a = \lim_{r\to\infty} e^a/r$) is

$$\chi = \begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\psi+\varphi)/2} & -\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\psi-\varphi)/2} \\ \gamma\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\psi-\varphi)/2} & \gamma\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\psi+\varphi)/2} \end{pmatrix} \chi_{(0)} , \tag{3.29}$$

where $\chi_{(0)}$ is any constant two-component spinor and we have defined

$$\gamma \equiv \mathrm{i}(2s + \sqrt{4s^2 - 1}) . \tag{3.30}$$

The Killing spinors in this 1/2 BPS class are thus given explicitly by (3.23), with $\chi$ given by (3.24), (3.29).

#### 1/4 BPS solutions

In this section we solve the Killing spinor equation (3.3) for the first class of BPS constraints (3.12). We again obtain an algebraic relation between $\epsilon_+$ and $\epsilon_-$ by using the integrability condition (D.1):

$$\begin{aligned} \epsilon_-^{(+)} &= \epsilon_+^{(+)} = 0 , \\ \epsilon_-^{(-)} &= \mathrm{i}\sqrt{\frac{r-s}{r+s}} \cdot \sqrt{\frac{(r-r_1)(r-r_2)}{(r-r_3)(r-r_4)}}\epsilon_+^{(-)} . \end{aligned} \tag{3.31}$$

Here we have identified the roots of $\Omega(r)$ in (3.15) as

$$\begin{aligned} \left\{ \begin{array}{c} r_4 \\ r_3 \end{array} \right\} &= s \pm \sqrt{\frac{2Q + 4s^2 - 1}{2}} , \\ \left\{ \begin{array}{c} r_2 \\ r_1 \end{array} \right\} &= -s \pm \sqrt{\frac{-2Q + 4s^2 - 1}{2}} . \end{aligned} \tag{3.32}$$

The $\mu = r$ component of the Killing spinor equation reads

$$\begin{aligned} \partial_r \,\epsilon_+ &= -\frac{\mathrm{i}}{2}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\,\epsilon_- - \mathrm{i}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\cdot\frac{1-2Q-4s^2}{4(r-s)^2}\,\tau_3\,\epsilon_- \ , \\ \partial_r \,\epsilon_- &= +\frac{\mathrm{i}}{2}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\,\epsilon_+ + \mathrm{i}\sqrt{\frac{r^2-s^2}{\Omega(r)}}\cdot\frac{1+2Q-4s^2}{4(r+s)^2}\,\tau_3\,\epsilon_+ \ . \end{aligned} \tag{3.33}$$

Using the relations (3.31) the general solution is

$$\epsilon = \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}} \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}} \end{pmatrix} \otimes \chi \ , \tag{3.34}$$

where again $\chi$ is a two-component spinor independent of $r$. The remaining components of equation (2.10) reduce to the following Killing spinor equation for $\chi$:

$$\left(\nabla^{(3)}_\alpha - \mathrm{i}A^{(3)}_\alpha + \frac{\mathrm{i}s}{2}\gamma_\alpha\right)\chi = 0 \ . \tag{3.35}$$

Here $\nabla^{(3)}_\alpha$ and $\gamma_\alpha$ are the spin connection and gamma matrices for the same biaxially squashed three-sphere metric (3.26), while (locally) the gauge field is now

$$A^{(3)} = \lim_{r\to\infty} A = P\sigma_3 = -\frac{1}{2}(4s^2-1)\sigma_3 \ . \tag{3.36}$$

Notice that (3.35) is *different* to the 1/2 BPS equation (3.25). The general solution to (3.35) in the orthonormal frame (3.28) is

$$\chi = \begin{pmatrix} 0 \\ \chi^{(-)}_{(0)} \end{pmatrix} \ , \tag{3.37}$$

where $\chi^{(-)}_{(0)}$ is a constant.

## 3.3 Regular self-dual Einstein solutions

Having completed the local analysis, in this section we continue by finding all globally regular supersymmetric Einstein solutions. These are necessarily self-dual, meaning that the Weyl tensor is self-dual, with the gauge field being an instanton, *i.e.* with self-dual field strength $F$.[6] The condition of regularity means requiring that the local metric given in (3.6) extends to a smooth complete metric on a four-manifold $\mathcal{M}^{(4)}$, and that the gauge field $A$ and Killing spinor are non-singular. Here it is important to specify globally precisely what are the gauge transformations of the gauge field $A$, and we shall find, throughout the whole paper, that regularity of the metric automatically

[6] Of course, a change of orientation replaces self-dual by anti-self-dual in these statements.

implies that $A$ satisfies the quantization condition for a *spin*$^c$ gauge field on $\mathcal{M}^{(4)}$, and that the Killing spinors are correspondingly then smooth spin$^c$ spinors.[7] We shall find two Einstein metrics in this class, both of which are known in the literature: the Taub-NUT-AdS solution, with the topology $\mathcal{M}^{(4)} = \mathbb{R}^4$ [78], and the Quaternionic-Eguchi-Hanson solutions, with topology the total space of the complex line bundle $\mathcal{M}^{(4)} = \mathcal{M}_p \equiv \mathcal{O}(-p) \to S^2$, for $p \geq 3$ [79, 80]. In fact these both derive from the *same* local solution in (3.6). These are not supersymmetric without the addition of an instanton gauge field.

### 3.3.1 BPS equations

It is straightforward to show that the metric in (3.6) is Einstein if and only if $P^2 - Q^2 = 0$. The field strength $F$ is then self-dual, meaning that the gauge field $A$ is an instanton. Thus, as commented in the previous section, the metrics in Class III are all Einstein. Recall that in this case

$$M = \mp s(4s^2 - 1) \ , \tag{3.38}$$

and the metric function $\Omega(r)$ in (3.7) simplifies to

$$\Omega(r) \quad = \quad (r \mp s)^2 [1 + (r \mp s)(r \pm 3s)] \ . \tag{3.39}$$

For the 1/2 BPS Class II, setting $P = \pm Q$ the BPS condition (3.13) implies

$$\text{1/2 BPS}: \qquad Q \quad = \quad \mp s\sqrt{4s^2 - 1} \ , \tag{3.40}$$

and hence again $M$ is given by (3.38). For the 1/4 BPS Class I, instead the BPS condition (3.12) gives

$$\text{1/4 BPS}: \qquad Q \quad = \quad \mp\frac{1}{2}(4s^2 - 1) \ , \tag{3.41}$$

which means that yet again $M$ is given by (3.38).

Thus for all cases with $P^2 = Q^2$ the metric is given by the *same* Einstein metric, with the metric function $\Omega(r)$ given by (3.39), but the gauge field instantons for the 1/2 BPS (3.40) and 1/4 BPS (3.41) classes are different. Class III clearly contains these supersymmetric solutions, but allows for an arbitrary rescaling of the instanton, described by the free parameter $P = \pm Q$. In fact we prove in appendix D.2 that the only *supersymmetric* solutions in Class III are the solutions above in Class I and II. We may thus henceforth discard Class III.

[7] In section 3.6 we shall discuss how uplifiting these solutions to *eleven dimensions* imposes further conditions, in particular it will turn out that $\lambda A$ is a *bona fide* connection, for some rational number $\lambda$ that we will determine. Correspondingly, the eleven-dimensional metric and Killing spinors will be globally defined only for certain choices of $p$, related to $\lambda$.

### 3.3.2 Einstein metrics

The Einstein metric described in the previous subsection is

$$\mathrm{d}s_4^2 \;=\; \frac{r^2-s^2}{\Omega(r)}\mathrm{d}r^2+(r^2-s^2)(\sigma_1^2+\sigma_2^2)+\frac{4s^2\Omega(r)}{r^2-s^2}\sigma_3^2\ , \tag{3.42}$$

where

$$\Omega(r) \;=\; (r\mp s)^2\left[1+(r\mp s)(r\pm 3s)\right]\ . \tag{3.43}$$

One can check that the Weyl tensor of this metric is self-dual. Notice that without loss of generality we may consider only the case $r\to+\infty$ for the asymptotic boundary (3.8). Due to the $\pm$ signs in (3.43) we may also without loss of generality assume that $s\geq 0$.[8]

It will be useful to note that the four roots of $\Omega(r)$ in (3.43) in this case may be written as

$$\begin{aligned}\begin{Bmatrix} r_4 \\ r_3 \end{Bmatrix} &= \pm s\ ,\\ \begin{Bmatrix} r_2 \\ r_1 \end{Bmatrix} &= \begin{Bmatrix} \mp s+\sqrt{4s^2-1} \\ \mp s-\sqrt{4s^2-1} \end{Bmatrix}\ .\end{aligned} \tag{3.44}$$

In particular, $r_1$ and $r_2$ are complex for $0\leq s<\frac{1}{2}$. Notice these agree with the corresponding limits of the general roots in (3.32); the relation to the roots in (3.21) is more complicated, and will be discussed in section 3.4.

#### Taub-NUT-AdS

We begin by considering the upper signs in (3.44). In this case $r_3=r_4=s$ is the largest root of $\Omega(r)$, so that $\Omega(r)>0$ for $r>s$. This case was discussed in [81], and the metric is automatically regular at the double root $r=s$ provided the Euler angle $\psi$ has period $4\pi$, so that the surfaces of constant $r>s$ are diffeomorphic to $S^3$. Then $\{r=s\}$ is a NUT-type coordinate singularity, and the metric is a smooth and complete metric on $\mathcal{M}^{(4)}=\mathbb{R}^4$, with the origin of $\mathbb{R}^4$ being naturally identified with $\{r=s\}$. In fact the metric is the metric on AdS$_4$ for the particular value $s=\frac{1}{2}$, with the limit $s=0$ being singular. The conformal boundary is correspondingly the round three-sphere for $s=\frac{1}{2}$, with $s>\frac{1}{2}$ and $0<s<\frac{1}{2}$ either "stretching" or "squashing" the size of the Hopf fibre $S^1$ relative to the $S^2$ base.

[8] At this point it might look more convenient to fix a choice of sign and simply take $s\in\mathbb{R}$. However, this choice of parametrization turns out to be inconvenient when comparing to the non-Einstein solutions discussed in later sections.

**Quaternionic-Eguchi-Hanson**

We next consider the lower signs in (3.44). In this case it is not possible to make the metric regular for $0 < s < \frac{1}{2}$, since in this range the largest root is at $r = -s < 0$, and the coefficient of $\sigma_3^2$ then blows up at $r = s > 0$, which leads to a singular metric. However, for $s > \frac{1}{2}$ the largest root is now at $r_2 = s + \sqrt{4s^2 - 1}$, and thus we might obtain a regular metric by taking $r \geq r_2$. To examine this possibility, we note that near to $r = r_2$ the metric is to leading order

$$\mathrm{d}s_4^2 \approx \frac{r_1 + s}{2(r - r_2)}\mathrm{d}r^2 + (r_2^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{8s^2(r - r_2)}{(r_1 + s)}\sigma_3^2 \ . \tag{3.45}$$

Changing coordinate to

$$R^2 = 2(r_1 + s)(r - r_2) \ , \tag{3.46}$$

the metric is to leading order near $R = 0$ given by

$$\mathrm{d}s_4^2 \approx \mathrm{d}R^2 + \left(\frac{2s}{r_1 + s}\right)^2 R^2(\mathrm{d}\psi + \cos\theta \mathrm{d}\varphi)^2 + (r_2^2 - s^2)(\sigma_1^2 + \sigma_2^2) \ . \tag{3.47}$$

We obtain a smooth metric on the $S^2$ at $R = 0$ provided that $\theta \in [0, \frac{\pi}{2}]$ and $\varphi$ has period $2\pi$. On surfaces $r > r_2$ we must then take $\psi$ to have period $4\pi/p$, so that these three-manifolds are biaxially squashed Lens spaces $S^3/\mathbb{Z}_p$. The collapse of the metric (3.47) at $R = 0$ is smooth if and only if the period $\Delta\psi = 4\pi/p$ of $\psi$ satisfies

$$\frac{2s}{r_1 + s}\Delta\psi = 2\pi \ . \tag{3.48}$$

We thus conclude that the squashing parameter is fixed to be

$$s = s_p \equiv \frac{p}{4\sqrt{p - 1}} \ . \tag{3.49}$$

Since $s > \frac{1}{2}$, this implies that for each integer $p \geq 3$ there exists a unique smooth Quaternionic-Eguchi-Hanson metric on the total space $\mathcal{M}_p$ of the complex line bundle $\mathcal{O}(-p) \to S^2$. In particular, the conformal boundary is then the biaxially squashed Lens space $S^3/\mathbb{Z}_p$, with squashing parameter fixed in terms of $p$ via (3.49).

The Quaternionc-Eguchi-Hanson metric is often presented in a different coordinate system. The change of variable

$$\begin{aligned} r(\rho)^2 &= s^2 + \frac{\rho - a^2}{(1 - \rho)^2} \ , \\ s^2 &= \frac{1}{2(1 - a^2)} \ , \end{aligned} \tag{3.50}$$

leads to the metric

$$\mathrm{d}s_4^2 \;=\; \frac{1}{(1-\rho)^2}\left[\frac{(\rho-a^2)\mathrm{d}\rho^2}{\rho^2-a^2}+(\rho-a^2)(\sigma_1^2+\sigma_2^2)+\frac{\rho^2-a^2}{\rho-a^2}\sigma_3^2\right] . \qquad (3.51)$$

In these coordinates the conformal boundary is at $\rho = 1$, and $a = a_p \equiv 1 - \frac{8(p-1)}{p^2}$.

### 3.3.3 Instantons

As already commented, the Taub-NUT-AdS and Quaternionic-Eguchi-Hanson manifolds are, by themselves, not supersymmetric. However, they become 1/2 BPS and 1/4 BPS solutions by turning on the instanton gauge field in (3.6) with $P = \pm Q$ and $Q$ fixed in terms of $s$ via (3.40) and (3.41), respectively. This is clear locally. In the remainder of this section we examine global issues. In particular, the instantons for the Quaternionic-Eguchi-Hanson solution will turn out to be automatically spin$^c$ connections in general, with the corresponding Killing spinor $\epsilon$ also being a spin$^c$ spinor. This is clearly necessary in order to have a smooth, globally-defined four-dimensional solution, since $\mathcal{M}_p \equiv \mathcal{O}(-p) \to S^2$ is a spin manifold if and only if $p$ is even, while it is spin$^c$ for all $p \in \mathbb{Z}$. We emphasize that in this section we are treating the solutions as purely four-dimensional. When we uplift to eleven-dimensional solutions in section 3.6 we will need to reconsider the gauge field $A$; in particular, what gauge transformations it inherits from eleven dimensions, and just as importantly whether it is $A$ that is “observable”, or rather some multiple of it – *cf.* footnote 7.

We begin by noting that with $P = \pm Q$ the *local* gauge field (3.6) is

$$A \;=\; P f_\pm(r)\sigma_3 \; , \qquad (3.52)$$

where we have defined

$$f_\pm(r) \;\equiv\; \frac{r \mp s}{r \pm s} \; . \qquad (3.53)$$

The corresponding field strength is thus

$$F \;=\; \mathrm{d}A \;=\; P f'_\pm(r)\mathrm{d}r\wedge\sigma_3 - P f_\pm(r)\sigma_1\wedge\sigma_2 \; . \qquad (3.54)$$

The value of $P$ is fixed to be

$$\begin{aligned} \text{1/4 BPS}: \qquad P \;&=\; -\frac{1}{2}(4s^2-1) \; , \\ \text{1/2 BPS}: \qquad P \;&=\; -s\sqrt{4s^2-1} \; . \end{aligned} \qquad (3.55)$$

**Taub-NUT-AdS**

Recall that for the Taub-NUT-AdS solution we must take the upper signs in (3.52). Then this gauge field is a globally well-defined one-form on $\{r > s\} \cong \mathbb{R}_+ \times S^3$. Crucially, at $r = s$ the function $f_+(s) = 0$. In fact near to this point $f_+(r)$ vanishes as $\rho^2$ as $\rho \to 0$, where $\rho$ denotes geodesic distance from the origin of $\mathbb{R}^4$ at $r = s$. It follows that $A$ is a global smooth one-form on the whole of $\mathcal{M}^{(4)} = \mathbb{R}^4$, and that the instanton is everywhere smooth and exact. This is true for either value of $P$ in (3.55). It follows that for all $s > 0$ we get a 1/2 BPS and a 1/4 BPS smooth Euclidean supersymmetric supergravity solution on $\mathbb{R}^4$. The 1/2 BPS solution was found in [81], while the 1/4 BPS solution is new.

**Quaternionic-Eguchi-Hanson**

Recall that for the Quaternionic-Eguchi-Hanson solution we must take the lower signs in (3.52). In this case the latter gauge field is not defined at $r = r_2$, where the vector field $\partial_\psi$ has zero length. However, the field stength (3.54) is manifestly a smooth global two-form on the four-manifold $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. It is straightforward to compute the flux through the $S^2 \subset \mathcal{M}_p$ at $r = r_2$:

$$\int_{S^2} \frac{F}{2\pi} \;=\; -2Pf_-(r_2) \;=\; \begin{cases} 4s^2 - 1 + 2s\sqrt{4s^2-1} & \text{1/4 BPS} \\ 4s^2 + 2s\sqrt{4s^2-1} & \text{1/2 BPS} \end{cases} , \tag{3.56}$$

where we have used (3.55). However, now using the fact that $s = s_p$ is fixed in terms of $p \geq 3$ via (3.49), we find the remarkable result

$$\int_{S^2} \frac{F}{2\pi} \;=\; \begin{cases} \frac{p}{2} - 1 & \text{1/4 BPS} \\ \frac{p}{2} & \text{1/2 BPS} \end{cases} . \tag{3.57}$$

In particular, for $p$ even we see that $F/2\pi$ defines an integral cohomology class in $H^2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$, while for $p$ odd instead $F/2\pi$ has *half-integer* period. This is precisely the condition that $A$ is a spin$^c$ connection. Recall that the curvature $F$ of a spin$^c$ connection $A$ on a manifold $\mathcal{M}$ satisfies the quantization condition

$$2\int_\Sigma \frac{F}{2\pi} \;=\; \int_\Sigma w_2(\mathcal{M}) \bmod 2 , \tag{3.58}$$

where $\Sigma \subset \mathcal{M}$ runs over all two-cycles in $\mathcal{M}$. Here $w_2(\mathcal{M}) \in H^2(\mathcal{M}, \mathbb{Z}_2)$ denotes the second Stiefel-Whitney class of (the tangent bundle of) $\mathcal{M}$. For $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$, it is straightforward to compute that $w_2(\mathcal{M}_p) = p \bmod 2 \in \mathbb{Z}_2 \cong H^2(\mathcal{M}_p, \mathbb{Z}_2)$. Thus for both 1/2 BPS and 1/4 BPS cases in (3.57) we see that $A$ is a spin$^c$ connection for *all* values of $p \geq 3$.

This is also clearly necessary for the Killing spinors in section 3.2.3 to be globally well-defined. For $p$ an odd integer, the manifolds $\mathcal{M}_p$ are *not* spin manifolds, so it is not possible to globally define a spinor $\epsilon$ on $\mathcal{M}_p$. However, from the Killing spinor equation (2.10) we see that $\epsilon$ is charged under the gauge field $A$. This precisely defines a spin$^c$ spinor, with spin$^c$ gauge field $A$, provided that the curvature $F = \mathrm{d}A$ satisfies the quantization condition (3.58). Thus the Killing spinors, in both 1/2 BPS and 1/4 BPS cases, are globally spin$^c$ spinors on $\mathcal{M}_p$. This is discussed in detail in appendix E. The upshot is that both the 1/2 BPS and 1/4 BPS Quaternionic-Eguchi-Hanson solutions on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$ lead to globally defined Euclidean supersymmetric supergravity solutions, for all $p \geq 3$. Specifically, the four-component Dirac (spin$^c$) spinors $\epsilon$ in the two cases are smooth sections of the bundles

$$\begin{cases} \pi^*[\mathcal{O}(p-2) \oplus \mathcal{O}(0) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(p)] & 1/4 \text{ BPS} \\ \pi^*[\mathcal{O}(p-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(p+1)] & 1/2 \text{ BPS} \end{cases}, \tag{3.59}$$

where $\pi : \mathcal{M}_p \to S^2$ denotes projection onto the bolt/zero-section.

We refer the reader to appendix E for a detailed discussion, but we conclude this section with some comments about the global form of the above Killing spinors and gauge field. In fact these comments will apply equally to all the four-dimensional solutions in this paper. The conformal boundary of the Quaternionic-Eguchi-Hanson solutions is a squashed $S^3/\mathbb{Z}_p$, with particular squashing fixed in terms of $p$ by (3.49). In the 1/2 BPS case the three-dimensional Killing spinor $\chi$ in (3.29) on constant $r > r_2$ hypersurfaces appears to depend on the coordinate $\psi$, but this is an artifact of the *frame* not being invariant under $\partial_\psi$. One can check that $\mathcal{L}_{\partial_\psi}\chi = 0$, and one is then free to take the $\mathbb{Z}_p$ quotient along $\psi$ and preserve supersymmety. When $p$ is odd the bulk spinors are necessarily spin$^c$ spinors, and these restrict to the unique spin bundle on the surfaces $\{r > r_2\} \cong S^3/\mathbb{Z}_p$. When $p$ is even the bulk is a spin manifold, and the surfaces $\{r > r_2\} \cong S^3/\mathbb{Z}_p$ have *two inequivalent* spin structures, which we refer to as "periodic" and "anti-periodic" in appendix E.[9] The spinor bundle of the bulk in fact restricts to the anti-periodic spinor bundle on $S^3/\mathbb{Z}_p$, but the spin$^c$ bundle in (3.59) that our Killing spinors are sections of restricts to the *periodic* spinor bundle on $S^3/\mathbb{Z}_p$. The $\frac{p}{2}$ units of flux in (3.57) play a crucial role in this discussion.

The 1/4 BPS case is essentially the same, but with one small difference. The three-dimensional Killing spinor $\chi$ in (3.37) appears to be independent of $\psi$, but now the rotating frame in fact means that $\mathcal{L}_{\partial_\psi}\chi = \frac{\mathrm{i}}{2}\chi$, introducing an overall $\psi$-dependence of $\mathrm{e}^{\mathrm{i}\psi/2}$ in $\chi$. Thus the 1/4 BPS spinors on $\{r > r_2\} \cong S^3/\mathbb{Z}_p$ hypersurfaces apparently depend on $\psi$, which would seem to prevent one from quotienting by $\mathbb{Z}_p$ and preserving supersymmetry. However, in solving the Killing spinor equation in section 3.2.3 we did

[9] This is by analogy with the two spin structures on $S^1$, but it is not meant to indicate any particular periodicity properties of the spinors.

not take into account the *global* form of the gauge field $A^{(3)}$. The full gauge field is

$$A^{(3)} = A^{(3)}_{\text{global}} + A^{(3)}_{\text{flat}} = P\sigma_3 + A^{(3)}_{\text{flat}} , \tag{3.60}$$

where $A^{(3)}_{\text{flat}}$ is a flat connection. The factor of $-1$ in the flux (3.57), relative to the 1/2 BPS case, precisely induces on $S^3/\mathbb{Z}_p$ a flat connection on the torsion line bundle $\mathscr{L}^{-1}$ with $c_1(\mathscr{L}) = 1 \in \mathbb{Z}_p \cong H^2(S^3/\mathbb{Z}_p, \mathbb{Z})$. The concrete effect of this is to introduce (locally) a phase $\mathrm{e}^{-\mathrm{i}\psi/2}$ into the Killing spinor $\chi$, cancelling the phase $\mathrm{e}^{\mathrm{i}\psi/2}$ described above, and meaning that the correct global form of the Killing spinor $\chi$ is in fact independent of $\psi$. Thus the $-1$ factor in (3.57), relative to the 1/2 BPS case, is crucial in order that these 1/4 BPS solutions are globally supersymmetric. We refer the interested reader to appendix E for a detailed discussion of these issues.

Finally, let us comment further on the global form of the boundary gauge field in (3.60). The gauge field at infinity $A^{(3)}$ is naively given by (3.52) restricted to $r = \infty$, which is

$$A^{(3)}_{\text{global}} \equiv P\sigma_3 , \tag{3.61}$$

where $\sigma_3$ is a globally defined one-form on $S^3/\mathbb{Z}_p$ (it is the global angular form for the fibration $S^3/\mathbb{Z}_p \to S^2$). Thus at first sight the gauge field at infinity is a global one-form, and thus is a connection on a trivial line bundle. However, this conclusion is false in general. The above argument is incorrect - the gauge field in (3.52) is defined only *locally* on $\mathcal{M}_p$, since it is ill-defined on the bolt at $r = r_2$, and for $p$ odd is not even globally a gauge field. This is discussed carefully in appendix E. If

$$\int_{S^2} \frac{F}{2\pi} = \frac{n}{2} , \tag{3.62}$$

then the upshot is that the gauge field at conformal infinity is (3.60) where $A^{(3)}_{\text{flat}}$ is a certain flat connection. Using the result of appendix E, we compute the first Chern class of the latter (which determines it uniquely) as

$$\mathbb{Z}_p \cong H^2(S^3/\mathbb{Z}_p, \mathbb{Z}) \ni c_1(A^{(3)}_{\text{flat}}) = \left\{ \begin{array}{ll} \left\{ \begin{array}{ll} \frac{p}{2} - 1 & p \text{ even} \\ p - 1 & p \text{ odd} \end{array} \right. & 1/4 \text{ BPS} \\ \left\{ \begin{array}{ll} \frac{p}{2} & p \text{ even} \\ 0 & p \text{ odd} \end{array} \right. & 1/2 \text{ BPS} \end{array} \right. . \tag{3.63}$$

Notice that the integers on the right hand side are defined only mod $p$. The term $P\sigma_3$ thus gives only the globally defined part of the gauge field, in general.

We conclude by emphasizing again that when we lift these solutions to eleven dimensions, in some cases we will need to re-examine the global form of the gauge transformations of $A$ inherited from eleven dimensions, to determine which solutions have the "same" boundary data. In particular, a flat gauge field such as $A^{(3)}_{\text{flat}}$ is always locally trivial, and the *only* information it contains is therefore global.

## 3.4 Regular 1/2 BPS solutions

In this section we find all globally regular supersymmetric solutions satisfying the 1/2 BPS condition (3.13). For all such solutions the (conformal class of the) boundary three-manifold will be $S^3/\mathbb{Z}_p$ with biaxially squashed metric

$$\mathrm{d}s^2_3 \;=\; \sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2 \;, \tag{3.64}$$

where $\sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi$ and $\psi$ has period $4\pi/p$, while the boundary gauge field is

$$A^{(3)} \;=\; P\sigma_3 + A^{(3)}_{\text{flat}} \;=\; -s\sqrt{4s^2-1}\sigma_3 + A^{(3)}_{\text{flat}} \;. \tag{3.65}$$

The flat gauge field $A^{(3)}_{\text{flat}}$ is present for precisely the same global reasons discussed at the end of section 3.3. The boundary Killing spinor equation is (3.25), which we reproduce here for convenience

$$\left(\nabla^{(3)}_\alpha - \mathrm{i}A^{(3)}_\alpha - \frac{\mathrm{i}s}{2}\gamma_\alpha - \frac{\mathrm{i}}{2}\sqrt{4s^2-1}\gamma_\alpha\gamma_3\right)\chi \;=\; 0 \;. \tag{3.66}$$

The solution $\chi$ is given by (3.29). It will be important to note that a solution to the above boundary data with given $s$ is diffeomorphic to the same solution with $s \to -s$. *Thus it is only $|s|$ that is physically meaningful at infinity.* This is completely obvious for the metric (3.64). We may effectively change the sign of $s$ in the gauge field (3.65) by the change of coordinates $\{\psi \to -\psi, \varphi \to -\varphi\}$, which sends $\sigma_3 \to -\sigma_3$. Similarly, we may effectively change the sign of $s$ in the Killing spinor equation (3.66) by sending $\gamma_\alpha \to -\gamma_\alpha$, which generate the same Clifford algebra Cliff$(3,0)$.

As we shall see, and perhaps surprisingly, for fixed conformal boundary data we sometimes find more than one smooth supersymmetric filling, with different topologies. This moduli space will be described in section 3.4.3.

### 3.4.1 Self-dual Einstein solutions

The 1/2 BPS Einstein solutions were described in section 3.3. For *any* choice of conformal boundary data, meaning for all $p \in \mathbb{N}$ and all choices of squashing parameter $s > 0$, there exists the 1/2 BPS Taub-NUT-AdS/$\mathbb{Z}_p$ solution on $\mathbb{R}^4/\mathbb{Z}_p$. This has metric (3.42), (3.43) and $\psi$ is taken to have period $4\pi/p$. This solution then has an isolated $\mathbb{Z}_p$ orbifold singularity at $r = s$ for $p > 1$, or, removing the singularity, the topology is $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p$. Although $\mathbb{R}^4/\mathbb{Z}_p$ is (mildly) singular for $p > 1$, there is evidence that this solution is indeed an appropriate gravity dual [82]. In the latter reference the large $N$ limit of the free energy of the ABJM theory on the unsquashed ($s = \frac{1}{2}$) $S^3/\mathbb{Z}_p$ was computed, and found to agree with the free energy of $\mathrm{AdS}_4/\mathbb{Z}_p$.

On the other hand, for each $p \geq 3$ and specific squashing parameter $s = s_p = \frac{p}{4\sqrt{p-1}}$ we also have the Quaternionic-Eguchi-Hanson solution. Thus for each $p \geq 3$ and $s = s_p$ there exist two supersymmetric self-dual Einstein fillings of the same boundary data: the Taub-NUT-AdS solution on $\mathbb{R}^4/\mathbb{Z}_p$ and the Quaternionic-Eguchi-Hanson solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. However, in concluding this we must be careful about the *global* boundary data in the two cases. As discussed around equation (3.63), the 1/2 BPS Quaternionic-Eguchi-Hanson solution has a gauge field on the conformal boundary $S^3/\mathbb{Z}_p$ with torsion first Chern class $c_1 = \frac{p}{2}$ mod $p$ when $p$ is even. That is, globally $A^{(3)}$ is a connection on the torsion line bundle $\mathscr{L}^{\frac{p}{2}}$ when $p$ is even, where $c_1(\mathscr{L}) = 1 \in \mathbb{Z}_p \cong H^2(S^3/\mathbb{Z}_p, \mathbb{Z})$ (notice $c_1 = 0 \bmod p$ when $p$ is odd). However, at the same time, the spinors in the bulk restrict to sections of the spin bundle $\mathscr{S}_1$ on the boundary. As discussed in detail in appendix E, in fact the latter bundle is isomorphic to $\mathscr{S}_0 \otimes \mathscr{L}^{\frac{p}{2}} \cong \mathscr{S}_1$, therefore the net effect of the non-trivial flat connection on the torsion line bundle $\mathscr{L}^{\frac{p}{2}}$ is to turn the boundary spinor into sections of $\mathscr{S}_0 \cong \mathscr{S}_1 \otimes \mathscr{L}^{\frac{p}{2}}$, the periodic spin bundle, precisely as for the spinors on the Taub-NUT-AdS solutions. Effectively, the additional flat gauge field induced from the bulk then cancels against the corresponding difference in the spin connection.

### 3.4.2 Non-self-dual Bolt solutions

#### Regularity analysis

We begin by analysing when the general metric in (3.6) is regular, where for the 1/2 BPS class the metric function $\Omega(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$ has roots[10]

$$\begin{aligned} \left\{ \begin{array}{c} r_4 \\ r_3 \end{array} \right\} &= \frac{1}{2}\left[-\sqrt{4s^2-1} \pm \sqrt{8s^2 - 4Q - 1}\right] \, , \\ \left\{ \begin{array}{c} r_2 \\ r_1 \end{array} \right\} &= \frac{1}{2}\left[\sqrt{4s^2-1} \pm \sqrt{8s^2 + 4Q - 1}\right] \, . \end{aligned} \tag{3.67}$$

Again, without loss of generality we may take the conformal boundary to be at $r = +\infty$. A complete metric will then necessarily close off at the largest root $r_0$ of $\Omega(r)$, which must satisfy $r_0 \geq s$ (if $r_0 < s$ then the metric (3.6) is singular at $r = s$). Given (3.67), the largest root is thus either $r_0 = r_+$ or $r_0 = r_-$, where

$$r_\pm \equiv \frac{1}{2}\left[\pm\sqrt{4s^2-1} + \sqrt{8s^2 \pm 4Q - 1}\right] \, . \tag{3.68}$$

[10] Notice that this parametrization of the roots is *different* to the self-dual Einstein limit in section 3.3.2. For example, setting $Q = -s\sqrt{4s^2-1}$ we have from (3.68) that $r_\pm = s$ for $s > 0$, which thus match onto the roots $r_3, r_4$ of section 3.3.2, while $r_\pm = -s \pm \sqrt{4s^2-1}$ for $s \leq -\frac{1}{2}$, which thus match onto the roots $r_1, r_2$ of section 3.3.2.

We first note that $r_0 = r_\pm = s$ leads only to the $Q = \mp s\sqrt{4s^2-1}$ Taub-NUT-AdS solutions considered in the previous section. Thus $r_0 > s$ and if $\psi$ has period $4\pi/p$ then the only possible topology is $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. Regularity of the metric near to the $S^2$ zero section at $r = r_0$ requires

$$\left|\frac{r_0^2 - s^2}{s\Omega'(r_0)}\right| \quad = \quad \frac{2}{p} \ . \tag{3.69}$$

This condition ensures that near to $\rho = 0$, where $\rho \equiv \lambda\sqrt{r - r_0}$ is geodesic distance near the bolt (for appropriate constant $\lambda > 0$), the metric (3.6) takes the form

$$\mathrm{d}s_4^2 \quad \approx \quad \mathrm{d}\rho^2 + \rho^2\left[\mathrm{d}\left(\frac{p\psi}{2}\right) + \frac{p}{2}\cos\theta\mathrm{d}\varphi\right]^2 + (r_0^2 - s^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \ . \tag{3.70}$$

Here $p\psi/2$ has period $2\pi$. Imposing (3.69) at $r_0 = r_\pm$ gives

$$Q \quad = \quad Q_\pm(s) \ \equiv \ \mp\frac{128s^4 - 16s^2 - p^2}{64s^2} \ . \tag{3.71}$$

In turn, one then finds that the putative largest root is

$$r_\pm(Q = Q_\pm(s)) \quad = \quad \frac{1}{8}\left[\frac{p}{|s|} \pm 4\sqrt{4s^2-1}\right] \ . \tag{3.72}$$

At this point we should pause to notice that a solution with given $s > 0$ will be equivalent to the corresponding solution with $s \to -s < 0$. This is because $Q_\pm(s) = Q_\pm(-s)$ in (3.71), which then leads to exactly the same set of roots in (3.67), and thus the same local metric, while $P(-s) = -P(s)$. However, from the explicit form of the gauge field in (3.6) we see that the diffeomorphism $\{\psi \to -\psi, \varphi \to -\varphi\}$ maps $\sigma_3 \to -\sigma_3$, which together with $s \to -s$ then leaves the gauge field invariant. Thus our parametrization of the roots in (3.67) is such that we need only consider $s > 0$, which we henceforth assume.

Recall that in order to have a smooth metric, we require $r_0 > s$. Imposing this for $r_0 = r_\pm(Q_\pm(s))$ gives

$$r_\pm(Q_\pm(s)) - s \quad = \quad f_p^\pm(s) \ , \tag{3.73}$$

where we must then determine the range of $s$ for which the function

$$f_p^\pm(s) \quad \equiv \quad \frac{1}{2}\left[\frac{p}{4s} - 2s \pm \sqrt{4s^2-1}\right] \tag{3.74}$$

is strictly positive, in order to have a smooth metric. In addition, we must verify that (3.72) really *is* the largest root. We thus define

$$r_\pm(Q_\pm(s)) - r_\mp(Q_\pm(s)) \quad = \quad h_p^\pm(s) \ , \tag{3.75}$$

where as in all other formulae in this paper the signs are read entirely along the top or the bottom, and one finds

$$h_p^{\pm}(s) \equiv \frac{1}{2}\left[\frac{p}{4s} \pm 2\sqrt{4s^2-1} - \sqrt{16s^2 - 2 - \frac{p^2}{16s^2}}\right] . \qquad (3.76)$$

Then (3.72) is indeed the largest root provided also $h_p^{\pm}(s)$ is positive, or is complex.[11]

We are thus reduced to determining the subset of $\{s > 0\}$ for which $f_p^{\pm}(s)$ is strictly positive, and $h_p^{\pm}(s)$ is either strictly positive or complex (since then the putative larger root is in fact complex). We refer to the two sign choices as positive and negative branch solutions. The behaviour for $p = 1$ and $p = 2$ is qualitatively different from that with $p \geq 3$, so we must treat these cases separately.

### $p = 1$

It is straightforward to see that $f_1^{\pm}(s) < 0$ for $s \in [\frac{1}{2}, \infty)$, so that the metric cannot be made regular for $s$ in this range. Specifically, $f_1^{\pm}(\frac{1}{2}) = -\frac{1}{4}$: since $f_1^{-}(s)$ is monotonic decreasing, this rules out taking $r_0 = r_-(Q_-(s))$ given by (3.72); on the other hand $f_1^{+}(s)$ monotonically increases to zero from below as $s \to \infty$, and we thus also rule out $r_0 = r_+(Q_+(s))$ in (3.72). For $s \in (0, \frac{1}{2})$ the putative largest root is complex, so this range is also not allowed. We thus conclude that there are no additional 1/2 BPS solutions with $p = 1$. This proves that the *only* 1/2 BPS solution with $S^3$ boundary is Taub-NUT-AdS.

### $p = 2$

We have $f_2^{\pm}(\frac{1}{2}) = 0$. Since $f_2^{-}(s)$ is monotonic decreasing on $s \in (\frac{1}{2}, \infty)$ we rule out the branch $r_0 = r_-(Q_-(s))$ for $s \in [\frac{1}{2}, \infty)$. On the other hand, one can check that $\frac{\mathrm{d}}{\mathrm{d}s} f_2^{+}(\frac{1}{2}) = +\infty$, $f_2^{+}(s)$ has a single turning point on $s \in (\frac{1}{2}, \infty)$ at $s = \frac{1}{4}\sqrt{2 + 2\sqrt{5}}$, and $f_2^{+}(s) \to 0$ from above as $s \to \infty$. In particular for all $s > \frac{1}{2}$ we may take $Q = Q_+(s)$ and $r_0(s) = r_+(Q_+(s))$, since we have shown that then $r_0(s) > s$ for all $s > \frac{1}{2}$. We must then check that $r_0(s)$ really is the largest root of $\Omega(r)$ in this range. This follows since $h_2^{+}(s) > 0$ holds for all $s$ in this range, and thus this positive branch exists for all $s > \frac{1}{2}$. Again, the roots are complex for $s \in (0, \frac{1}{2})$. In conclusion, we have shown that for all $s \in (\frac{1}{2}, \infty)$ we have a regular 1/2 BPS solution on $\mathcal{M}_2 = \mathcal{O}(-2) \to S^2$.[12]

[11] If $h_p^{\pm}(s)$ is negative, one cannot then simply take the larger root $r_{\mp}(Q_{\pm}(s))$ to be $r_0$, as the regularity condition (3.69) does not hold.

[12] Notice that the $s = \frac{1}{2}$ limiting solution fills a *round* Lens space $S^3/\mathbb{Z}_2$. We shall discuss this further in section 3.4.2.

$p \geq 3$

**Positive branches**

One can check that for all $p > 2$ we have $f_p^+(\frac{1}{2}) > 0$, $\frac{\mathrm{d}}{\mathrm{d}s} f_p^+(\frac{1}{2}) = +\infty$, and $f_p^+(s)$ has a single turning point on $s \in (\frac{1}{2}, \infty)$ given by

$$\frac{\mathrm{d}}{\mathrm{d}s} f_p^+(s) \;=\; 0 \qquad \Longrightarrow \qquad s \;=\; \sqrt{\frac{p(4-p+\sqrt{p(p+8)})}{32(p-1)}} \;. \tag{3.77}$$

Moreover, then $f_p^+(s) \to 0$ from above as $s \to \infty$. Setting $Q = Q_+(s)$, we must check that $r_+(Q_+(s))$ is the largest root. In fact $r_-(Q_+(s))$, and hence $h_p^+(s)$, is real here only for $s \geq \frac{1}{4}\sqrt{1+\sqrt{p^2+1}}$. In this range (which notice is automatic when $p = 2$) one can check that $h_p^+(s)$ is strictly positive. In conclusion, taking $Q = Q_+(s)$ one finds that $r_0(s) = r_+(Q_+(s))$ is indeed the largest root of $\Omega(r)$ and satisfies $r_0(s) > s$ for all $s \geq \frac{1}{2}$. Thus the metric is regular. In conclusion, we have shown that for all $s \in [\frac{1}{2}, \infty)$ we have a regular 1/2 BPS solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$.

**Negative branches**

For $p \geq 3$ we also have regular solutions from the negative branch. Indeed, we now have $f_p^-(\frac{1}{2}) > 0$. Since $f_p^-(s)$ is monotonically decreasing, it follows that $f_p^-(s)$ is positive on precisely $[\frac{1}{2}, s_p)$ for some $s_p > 1$. One easily finds

$$s_p \;=\; \frac{p}{4\sqrt{p-1}} \;. \tag{3.78}$$

Again, notice here that $p = 2$ is special, since $s_2 = \frac{1}{2}$. There is thus potentially another branch of solutions for $s$ in the range

$$\frac{1}{2} \;\leq\; s \;<\; \frac{p}{4\sqrt{p-1}} \;=\; s_p \;. \tag{3.79}$$

To check this is indeed the case, we note that $h_p^-(s)$ is real only for $s \geq \frac{1}{4}\sqrt{1+\sqrt{p^2+1}}$, and one can check that provided also $s < s_p$ then $h_p^-(s)$ is positive. Thus $r_-(Q_-(s))$ is indeed the largest root of $\Omega(r)$ for $Q = Q_-(s)$ and $s$ satisfying (3.79). In conclusion, we have shown that for all $s \in [\frac{1}{2}, s_p)$ we have a regular 1/2 BPS solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. The limiting solutions for $s = s_p$, which notice are where the roots $r_\pm(Q_-(s))$ are equal, will be discussed later.

**Gauge field and spinors**

Having determined this rather intricate branch structure of solutions, let us now turn to analysing the global properties of the gauge field. After a suitable gauge transformation, the latter can be written *locally* as

$$A = \frac{s}{r^2 - s^2}\left[-2Qr - (r^2 + s^2)\sqrt{4s^2 - 1}\right]\sigma_3 \ . \tag{3.80}$$

In particular, this gauge potential is *singular* on the $S^2$ at $r = r_0$, but is otherwise globally defined on the complement $\mathcal{M}_p \setminus S^2$ of the bolt. The field strength $F = \mathrm{d}A$ is easily verified to be a globally defined smooth two-form on $\mathcal{M}_p$, with non-trivial flux through the $S^2$ at $r = r_0$. Indeed, for $Q = Q_\pm(s)$ one computes the period through the $S^2$ at $r_0(s) = r_\pm(Q_\pm(s))$ (respectively) to be

$$\begin{aligned}\int_{S^2}\frac{F}{2\pi} &= -\frac{2s}{r_0(s)^2 - s^2}\left[-2Q_\pm(s)r_0(s) - (r_0(s)^2 + s^2)\sqrt{4s^2 - 1}\right] \\ &= \pm\frac{p}{2}\ ,\end{aligned} \tag{3.81}$$

the last line simply being a remarkable identity satisfied by the largest root $r_0(s)$. Thus the positive/negative branch solutions have a gauge field flux $\pm\frac{p}{2}$ through the bolt, respectively. Following appendix E, and precisely as for the 1/2 BPS Quaternionic-Eguchi-Hanson solutions in section 3.3.3, both branches then induce the *same* spinors and global gauge field at conformal infinity, for fixed $p$ and $s$ (the crucial point here being that $\frac{p}{2} \equiv -\frac{p}{2}$ mod $p$, so that the torsion line bundles on the boundary are the same for the positive and negative branches). Again, in eleven dimensions we will need to reconsider this conclusion, as the physically observable gauge field is not necessarily $A$, but rather a multiple of it.

For completeness we note that the Dirac spin$^c$ spinors are smooth sections of the following bundles:

$$\begin{cases} \pi^*\left[\mathcal{O}(p-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(p+1)\right] & \text{positive branch} \\ \pi^*\left[\mathcal{O}(-1) \oplus \mathcal{O}(-p+1) \oplus \mathcal{O}(-p-1) \oplus \mathcal{O}(1)\right] & \text{negative branch}\end{cases}\ , \tag{3.82}$$

and that when $p$ is even the boundary gauge field $A^{(3)}$ is a connection on $\mathscr{L}^{\frac{p}{2}}$.

#### Special solutions

For $p \geq 2$ the positive branches described in section 3.4.2 all terminate at $s = \frac{1}{2}$, while for $p \geq 3$ the negative branches terminate at $s = \frac{1}{2}$ and $s = s_p$. In this section we consider these special limiting solutions.

#### Positive branches

When $s = \frac{1}{2}$ note firstly that the conformal boundary $S^3/\mathbb{Z}_p$ is *round*, and secondly that the *global* part of the gauge field $A^{(3)}_{\text{global}}$ on the conformal boundary is identically zero.

Indeed, notice that $P = 0$ when $s = \frac{1}{2}$, while

$$Q = Q_+\left(\tfrac{1}{2}\right) = \frac{(p-2)(p+2)}{16} . \tag{3.83}$$

Thus for $p = 2$ in particular we see that $P = 0 = Q$ and thus this solution is self-dual, but with a round $S^3/\mathbb{Z}_2$ boundary. It is not surprising, therefore, to discover that $s = \frac{1}{2}$ is simply $\mathrm{AdS}_4/\mathbb{Z}_2$ in this case. However, due to the single unit of gauge field flux through the bolt (which in this singular limit has collapsed to zero size), the global gauge field on the boundary is the unique non-trivial flat $U(1)$ connection on $S^3/\mathbb{Z}_2$.[13]

For $p \geq 3$ we also have $P = 0$, but now $Q > 0$ in (3.83). Thus the gauge field in the bulk is *not* an instanton, and correspondingly we obtain a non-trivial smooth non-self-dual solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. We will refer to all these solutions as *round Lens filling* solutions – locally, the conformal boundary is equivalent to the round three-sphere.

Although this branch does not terminate at $s = s_p$, we note that at this point $Q_+(s_p) = s_p\sqrt{4s_p^2 - 1} = -P$ so that the solution is *self-dual*. In fact this solution is precisely the Quaternionic-Eguchi-Hanson solution! Thus although this was isolated as a self-dual solution, we see that it exists as a special case of a family of non-self-dual solutions.

#### Negative branches

The discussion for the limit $s = \frac{1}{2}$ is similar to that for the positive branches above. The only difference is that now

$$Q = Q_-\left(\tfrac{1}{2}\right) = -\frac{(p-2)(p+2)}{16} . \tag{3.84}$$

However, since $P = 0$ and $r_+(Q_+(\frac{1}{2})) = r_-(Q_-(\frac{1}{2}))$, we see that these are actually the *same* round Lens filling solutions as on the positive branch. Thus the positive and negative branches actually *join together* at this point.

Finally, recall that the $s = s_p$ limit has $h_p^-(s_p) = 0$, implying that we have a *double root*. It follows that this must locally be a Taub-NUT-AdS solution, and indeed one can check that this negative branch joins onto Taub-NUT-AdS$/\mathbb{Z}_p$ with squashing parameter $s = s_p$.

### 3.4.3 Moduli space of solutions

We have summarized the intricate branch structure of solutions in Figure 3.1. In general the conformal boundary has biaxially squashed $S^3/\mathbb{Z}_p$ metric (3.64), with squashing parameter $s > 0$, and boundary gauge field given by (3.65). The 1/2 BPS fillings of this boundary may then be summarized as follows:

[13] Correspondingly, the spinors inherited from the bulk are sections of $\mathscr{S}_1$, so that altogether the boundary spinors are sections of $\mathscr{S}_0$.

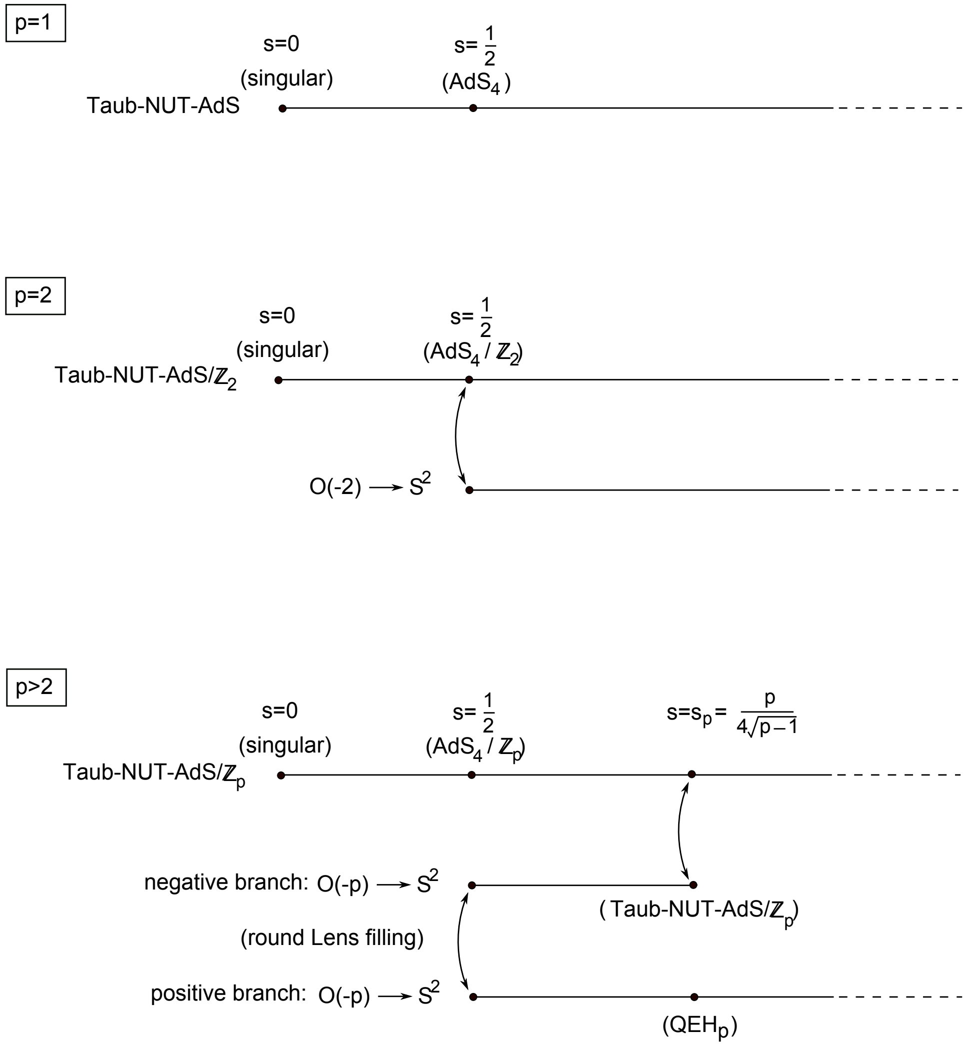


Figure 3.1: The moduli space of 1/2 BPS solutions with biaxially squashed $S^3/\mathbb{Z}_p$ boundary, with squashing parameter $s$. The arrows denote identification of solutions on different branches. Notice that these moduli spaces are connected for each $p$, but that for $p \geq 2$ the space multiply covers the $s$-axis. The self-dual Quaternionic-Eguchi-Hanson solution $\mathrm{QEH}_p$ appears as a special point on the positive branch for $p \geq 3$.

- For $p = 1$, the boundary $S^3$ with arbitrary squashing parameter $s > 0$ has a unique 1/2 BPS filling, namely the Taub-NUT-AdS solution. For $s = \frac{1}{2}$ one obtains the $\mathrm{AdS}_4$ metric as a special case. The gauge field curvature is real for $s > 1/2$ and imaginary for $s < 1/2$.

- For $p \geq 2$ and arbitrary squashing parameter $s > 0$ we always have the (mildly singular) Taub-NUT-AdS/$\mathbb{Z}_p$ solution. Thus for all boundary data there always exists a gravity filling, provided one allows for orbifold singularities. However, starting with $p = 2$ there can exist other 1/2 BPS solutions, leading to non-unique supersymmetric fillings of the same boundary:

- For $p = 2$ and $s > \frac{1}{2}$ there is also a 1/2 BPS filling with the topology $\mathcal{M}_2 = \mathcal{O}(-2) \to S^2$. This degenerates to $\mathrm{AdS}_4/\mathbb{Z}_2$ in the $s \to \frac{1}{2}$ limit, but with a non-trivial flat connection. This solution was first found in [81], where it was dubbed supersymmetric Eguchi-Hanson-AdS. Notice that for $p = 2$ and $s = \frac{1}{2}$ there then exists a unique filling of the round $S^3/\mathbb{Z}_2$, which is the singular $\mathrm{AdS}_4/\mathbb{Z}_2$ solution.
- For all $p > 2$ and $s > \frac{1}{2}$ there is an even more intricate structure. There is always a positive branch filling with topology $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$, which includes the Quaternionic-Eguchi-Hanson solution at the specific value $s = s_p = \frac{p}{4\sqrt{p-1}}$. In the $s = \frac{1}{2}$ limit (which is non-singular) this branch joins onto a negative branch set of solutions, with the same topology. However, this negative branch then exists only for $s < s_p$, and joins onto the Taub-NUT-AdS/$\mathbb{Z}_p$ general solutions in the $s \to s_p$ limit. In particular, notice that this moduli space is connected, but multiply covers the $s$-axis.

### 3.4.4 Holographic free energy

In this subsection we compute the holographic free energy of the 1/2 BPS solutions summarized above, using standard holographic renormalization methods [50, 51]. Further details can be found in appendix F.1. A subtlety for $p > 1$ is how to calculate the holographic free energy of the *singular* Taub-NUT-AdS/$\mathbb{Z}_p$ solutions, that we shall discuss later.

The total on-shell action is

$$I \;=\; I^{\text{grav}}_{\text{bulk}} + I^F + I^{\text{grav}}_{\text{ct}} + I^{\text{grav}}_{\text{bdry}} \,. \tag{3.85}$$

Here the first two terms are the bulk supergravity action (3.1)

$$I^{\text{grav}}_{\text{bulk}} + I^F \;\equiv\; -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\left(R + 6 - F^2\right) \,, \tag{3.86}$$

evaluated on a particular solution. This is divergent, but we may regularize it using holographic renormalization. Introducing a cut-off at some large value of $r = \varrho$, with corresponding hypersurface $\mathcal{S}_\varrho = \{r = \varrho\}$, we then add the following boundary terms

$$I^{\text{grav}}_{\text{ct}} + I^{\text{grav}}_{\text{bdry}} \;=\; \frac{1}{8\pi G_4}\int_{\mathcal{S}_\varrho} \mathrm{d}^3x\sqrt{\gamma}\left(2 + \tfrac{1}{2}R(\gamma) - K\right) \,. \tag{3.87}$$

Here $R(\gamma)$ is the Ricci scalar of the induced metric $\gamma_{\mu\nu}$ on $\mathcal{S}_\varrho$, and $K$ is the trace of the second fundamental form of $\mathcal{S}_\varrho$, the latter being the Gibbons-Hawking boundary term.

In all cases the manifold closes off at $r = r_0$, the largest root of $\Omega(r)$, and we compute

$$
\begin{aligned}
I^{\text{grav}}_{\text{bulk}} &= \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\left(2sr^3 - 6s^3 r\right)\Big|_{r_0}^{\varrho} , && (3.88)\\
I^{\text{grav}}_{\text{ct}} + I^{\text{grav}}_{\text{bdry}} &= \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\left[2Qs\sqrt{4s^2-1} - 2s\varrho^3 + 6s^3\varrho + O(\varrho^{-1})\right] . && (3.89)
\end{aligned}
$$

As expected, the divergent terms cancel as $\varrho \to \infty$. The contribution to the action of the gauge field is finite in all cases and does not need regularization. For the Taub-NUT-AdS case $r_0 = s$ and we compute

$$
I^F_{\text{NUT}} = \frac{16\pi^2}{8\pi G_4}Q^2 = \frac{2\pi}{G_4}s^2(4s^2-1) \qquad (p=1) , \tag{3.90}
$$

while for the Taub-Bolt-AdS cases $r_0 = r_\pm > s$ and we compute

$$
I^F_{\text{Bolt}} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\frac{2sr_0\left[\left(Qr_0 + s^2\sqrt{4s^2-1}\right)^2 + \left(Qs + sr_0\sqrt{4s^2-1}\right)^2\right]}{(r_0^2 - s^2)^2} . \tag{3.91}
$$

Combining all the above contributions to the action we obtain the following simple expressions

$$
\begin{aligned}
I_{\text{NUT}} &= 2s^2\frac{\pi}{G_4} \qquad (p=1) ,\\
I_{\text{Bolt}_\pm} &= \left[\frac{1}{2} \pm \frac{\sqrt{4s^2-1}}{sp}\left(s^2 - \frac{p^2}{16}\right)\right]\frac{\pi}{G_4} .
\end{aligned} \tag{3.92}
$$

Here $I_{\text{Bolt}_\pm}$ refers to the actions of the positive and negative branch solutions, respectively. Recall that $I_{\text{Bolt}_+}$ exists[14] for any $p \geq 2$, while $I_{\text{Bolt}_-}$ exists for any $p \geq 3$.

For any $p \geq 2$ we can always fill the boundary squashed Lens space $S^3/\mathbb{Z}_p$ with the mildly singular Taub-NUT-AdS/$\mathbb{Z}_p$ solution, where $\mathbb{Z}_p$ acts on the coordinate $\psi$. In these cases one may be concerned that the supergravity approximation breaks down and the classical on-shell gravity action (3.85) does not reproduce the correct free energy of the holographic dual field theories. In particular, the fact that the Taub-Bolt-AdS solutions smoothly reduce to the Taub-NUT-AdS/$\mathbb{Z}_p$ solutions at the special points $(p=2, s=\frac{1}{2})$ and $(p \geq 3, s = s_p)$ (see Figure 3.1) implies that the holographic free energies of these orbifold solutions must be given by the limits

$$
\begin{aligned}
\lim_{s\to\frac{1}{2}} I_{\text{Bolt}_+} &= \frac{1}{2}\frac{\pi}{G_4} \qquad (p=2) ,\\
\lim_{s\to s_p} I_{\text{Bolt}_-} &= \frac{p^2}{8(p-1)}\frac{\pi}{G_4} \qquad (p \geq 3) ,
\end{aligned} \tag{3.93}
$$

respectively. These differ from the naive values $\frac{1}{p}I_{\text{NUT}}$ of the Taub-NUT-AdS/$\mathbb{Z}_p$ solu-

[14] For $p = 2$ this free energy was computed in [81].

tions by a contribution that can be understood as associated to flux trapped at the $\mathbb{Z}_p$ singularity [81]. In turn, this trapped flux is related directly to the fact that the Taub-NUT-AdS/$\mathbb{Z}_p$ limits of the Taub-Bolt-AdS solutions necessarily have an additional flat gauge field $A^{(3)}_{\text{flat}}$ turned on, relative to the simple $\mathbb{Z}_p$ quotient of the $p = 1$ Taub-NUT-AdS solution. In similar circumstances (*e.g.* in singular ALE Calabi-Yau two-folds), a method for computing the contribution of this flux is to resolve the space. However, presently we cannot resolve the space while preserving supersymmetry (and $SU(2) \times U(1)$ isometry), as such geometries would contain two parameters and their existence is precluded by our general analysis. It is natural to assume that, by continuity, the free energy of the orbifold Taub-NUT-AdS/$\mathbb{Z}_p$ branch onto which the bolt solutions join contains the contribution of this trapped flux for generic values of $s$. One way to compute the free energies of these solutions is to resolve the NUT orbifold singularity, replacing it with a non-vanishing two-sphere $S^2_\varepsilon$, while not preserving supersymmetry. Using this method, further discussed in appendix F.2, we find that for a gauge field with $\frac{n}{2}$ units of flux at the singularity the contribution to the free energy is given by

$$I_{\text{sing}} = \frac{n^2}{8p} \cdot \frac{\pi}{G_4} . \tag{3.94}$$

The total free energy of the orbifold solutions with $\pm\frac{p}{2}$ units of flux is then given by

$$I^{\text{orb}}_{\text{NUT+flux}} = \frac{1}{p} I_{\text{NUT}} + I_{\text{sing}} = \left( \frac{2s^2}{p} + \frac{p}{8} \right) \frac{\pi}{G_4} . \tag{3.95}$$

In Figure 3.2 we have plotted the holographic free energies for various values of $p$. The first plot is the free energy of the unique 1/2 BPS filling of the squashed $S^3$, with the marked point being the AdS$_4$ solution without gauge field. In the second plot $p = 2$ and we see that the free energy of the positive branch bolt solution joins at $s = \frac{1}{2}$ to the free energy of the orbifold Taub-NUT-AdS/$\mathbb{Z}_2$ solution with 1 unit of flux at the singularity, as observed in [81]. On the same plot the green curve is the free energy of Taub-NUT-AdS/$\mathbb{Z}_2$, without any trapped flux. In the remaining two plots ($p = 5$ and $p = 12$ respectively) the negative branch bolt solutions appear. The curve of the free energy $I_{\text{Bolt}_-}$ connects the free energy $I^{\text{orbifold}}_{\text{NUT+flux}}$ of the orbifold branch with the free energy $I_{\text{Bolt}_+}$ of the positive branch at the values $s = s_p$ and $s = \frac{1}{2}$, respectively.

## 3.5 Regular 1/4 BPS solutions

In this section we find all regular supersymmetric solutions satisfying the 1/4 BPS condition (3.12). For all solutions the (conformal class of the) boundary three-manifold is again a biaxially squashed $S^3/\mathbb{Z}_p$ with metric (3.64), but now the boundary gauge field

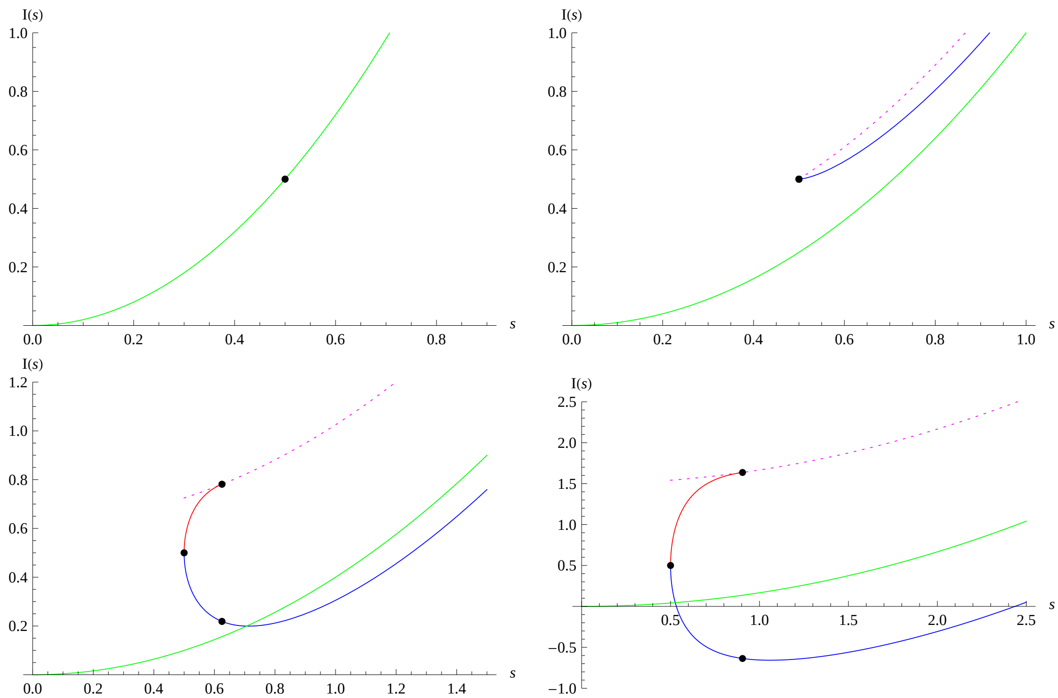


Figure 3.2: Plots of the free energies $I(s)$ of the different branches for $p = 1, 2, 5, 12$, respectively. The first plot is the free energy of the 1/2 BPS Taub-NUT-AdS solution. In the other plots the green curve is the free energy $\frac{1}{p} I_{\mathrm{NUT}}$ of the Taub-NUT-AdS/$\mathbb{Z}_p$ solution, while the dotted line in magenta is the free energy $I^{\mathrm{orb}}_{\mathrm{NUT+flux}}$, including the contribution of $\pm\frac{p}{2}$ units of flux at the orbifold singularity. The red curve is the free energy $I_{\mathrm{Bolt}_-}$ of the negative branch. The blue curve is the free energy $I_{\mathrm{Bolt}_+}$ of the positive branch. The free energies of the special solutions are marked with points.

is given by

$$A^{(3)} \;=\; P\sigma_3 + A^{(3)}_{\mathrm{flat}} \;=\; -\frac{1}{2}(4s^2-1)\sigma_3 + A^{(3)}_{\mathrm{flat}}\ , \tag{3.96}$$

where $A^{(3)}_{\mathrm{flat}}$ is again a certain flat connection. The latter is particularly important in order to globally have supersymmetry on the boundary in this case, precisely as for the 1/4 BPS Quaternionic-Eguchi-Hanson solutions in section 3.3.3. The boundary Killing spinor equation is (3.35), which we reproduce here for convenience:

$$\left(\nabla^{(3)}_\alpha - \mathrm{i}A^{(3)}_\alpha + \frac{\mathrm{i}s}{2}\gamma_\alpha\right)\chi \;=\; 0\ . \tag{3.97}$$

Again as in section 3.4, a solution to the above boundary data with given $s$ is diffeomorphic to the same solution with $s \to -s$.

As for the case of 1/2 BPS solutions, for fixed conformal boundary data we find more than one smooth supersymmetric filling, with different topologies. What is exceptional in the 1/4 BPS class of solutions is that for an $S^3$ boundary the Taub-NUT-AdS solution is *not* the unique filling, as one might expect, but rather there is also a filling with an

$\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$ topology. The full moduli space will be summarized in section 3.5.3.

### 3.5.1 Self-dual Einstein solutions

The 1/4 BPS Einstein solutions were described in section 3.3. For *any* choice of conformal boundary data, meaning for all $p \in \mathbb{N}$ and all choices of squashing parameter $s > 0$, there exists the 1/4 BPS Taub-NUT-AdS solution on $\mathbb{R}^4/\mathbb{Z}_p$. This has metric (3.42), (3.43) and $\psi$ is taken to have period $4\pi/p$. This solution then has an isolated $\mathbb{Z}_p$ orbifold singularity at $r = s$ for $p > 1$, or, removing the singularity, the topology is $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p$. In taking the $\mathbb{Z}_p$ quotient in this 1/4 BPS case notice that in order to preserve supersymmetry we must also turn on an additional flat gauge field which is a connection on $\mathscr{L}^{-1}$. Here recall that $\mathscr{L}$ is the line bundle on $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p$ with torsion first Chern class $c_1(\mathscr{L}) = 1 \in H^2(\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p, \mathbb{Z}) \cong \mathbb{Z}_p$. The reason for this is as discussed for the Quaternionic-Eguchi-Hanson solutions in section 3.3.3 – the Killing spinors for the 1/4 BPS Taub-NUT-AdS solution are not invariant under $\mathcal{L}_{\partial_\psi}$, and the additional torsion gauge field is *required* in order to have supersymmetry on the quotient space.

On the other hand, for each $p \geq 3$ and specific squashing parameter $s = s_p = \frac{p}{4\sqrt{p-1}}$ we also have the 1/4 BPS Quaternionic-Eguchi-Hanson solution. Thus for each $p \geq 3$ and $s = s_p$ there exist two supersymmetric self-dual Einstein fillings of the same boundary data: the Taub-NUT-AdS solution on $\mathbb{R}^4/\mathbb{Z}_p$ and the Quaternionic-Eguchi-Hanson solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. Again, the boundary gauge field is important in comparing the *global* boundary data for these two solutions, and the discussion is essentially the same as for the 1/2 BPS case in section 3.4.1. In fact the only difference between the two cases is the additional contribution of $\mathscr{L}^{-1}$ described in the previous paragraph.

### 3.5.2 Non-self-dual Bolt solutions

#### Regularity analysis

We begin by analysing when the general metric in (3.6) is regular, where for the 1/4 BPS class the metric function $\Omega(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$ has roots

$$\begin{aligned} \left\{ \begin{array}{c} r_4 \\ r_3 \end{array} \right\} &= s \pm \sqrt{\frac{-1 + 2Q + 4s^2}{2}} \, , \\ \left\{ \begin{array}{c} r_2 \\ r_1 \end{array} \right\} &= -s \pm \sqrt{\frac{-1 - 2Q + 4s^2}{2}} \, . \end{aligned} \tag{3.98}$$

Again, without loss of generality we may take the conformal boundary to be at $r = +\infty$. A complete metric will then necessarily close off at the largest root $r_0$ of $\Omega(r)$, which must satisfy $r_0 \geq s$. Given (3.98), the largest root is thus either $r_0 = r_+$ or $r_0 = r_-$, where

$$r_\pm \quad \equiv \quad \pm s + \sqrt{\frac{-1 \pm 2Q + 4s^2}{2}} \ . \tag{3.99}$$

We first note that $r_0 = r_\pm = s$ leads only to the $Q = \mp\frac{1}{2}(4s^2 - 1)$ Taub-NUT-AdS solutions considered in the previous section. Thus $r_0 > s$ and if $\psi$ has period $4\pi/p$ then the only possible topology is $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. Regularity of the metric near to the $S^2$ zero section at $r = r_0$ requires, as in the previous section,

$$\left|\frac{r_0^2 - s^2}{s\Omega'(r_0)}\right| \quad = \quad \frac{2}{p} \ . \tag{3.100}$$

Imposing (3.100) at $r_0 = r_+$ gives

$$Q \ = \ \begin{cases} Q_+(s) \ , & s > 0 \\ \\ -Q_-^\pm(s) \ , & s < 0 \end{cases} \ , \tag{3.101}$$

while for $r_0 = r_-$ imposing (3.100) gives

$$Q \ = \ \begin{cases} Q_-^\pm(s) \ , & s > 0 \\ \\ -Q_+(s) \ , & s < 0 \end{cases} \ . \tag{3.102}$$

Here we have defined

$$\begin{aligned} Q_+(s) \quad &\equiv \quad \frac{p^2 - (16s^2 - p)\sqrt{f_p^+(s)}}{128s^2} \ , \\ Q_-^\pm(s) \quad &\equiv \quad -\frac{p^2 \mp (16s^2 + p)\sqrt{f_p^-(s)}}{128s^2} \ , \end{aligned} \tag{3.103}$$

and have introduced the polynomials

$$f_p^\pm(s) \ \equiv \ (16s^2 \pm p)^2 - 128s^2 \ . \tag{3.104}$$

Similarly to the 1/2 BPS solutions, notice that a solution with given $s > 0$ will be equivalent to the corresponding solution with $s \to -s < 0$. This is because $r_+(s) = r_-(-s)$, which then leads to exactly the same set of roots in (3.98), and thus the same local metric. In addition, $P(-s) = P(s)$ and $Q_\pm(s) = Q_\pm(-s)$ and hence the gauge field in (3.6) is also invariant. Thus our parametrization of the roots in (3.98) is such that we need only consider $s > 0$, which we henceforth assume.

The putative largest root for (3.101) and (3.102), respectively, is

$$
\begin{aligned}
r_+(Q = Q_+(s)) &= \frac{p + \sqrt{f_p^+(s)}}{16s} \, , \\
r_-(Q = Q_-^{\pm}(s)) &= \frac{p \mp \sqrt{f_p^-(s)}}{16s} \, .
\end{aligned}
\tag{3.105}
$$

The above expressions are real provided $f_p^{\pm}(s)$ are positive semidefinite.

Recall that in order to have a smooth metric we require $r_0 > s$. Imposing this for $r_0 = r_+(Q_+(s))$ and $r_0 = r_-(Q_-^{\pm}(s))$ is equivalent to determining the range of $s$ for which the functions

$$
\begin{aligned}
a_p(s) &\equiv (p - 16s^2) + \sqrt{f_p^+(s)} \, , \\
b_p^{\pm}(s) &\equiv (p - 16s^2) \mp \sqrt{f_p^-(s)} \, ,
\end{aligned}
\tag{3.106}
$$

are strictly positive, respectively. In addition, we must verify that (3.105) really *is* the largest root. We thus define

$$
\begin{aligned}
c_p(s) &\equiv r_+(Q_+(s)) - r_-(Q_+(s)) \, , \\
d_p^{\pm}(s) &\equiv r_-(Q_-^{\pm}(s)) - r_+(Q_-^{\pm}(s)) \, ,
\end{aligned}
\tag{3.107}
$$

and one finds

$$
c_p(s) = \frac{p + 16s^2 + \sqrt{f_p^+} - \sqrt{(p - 16s^2 - \sqrt{f_p^+})^2 - 4p^2}}{16s} \, , \tag{3.108}
$$

$$
d_p^{\pm}(s) = \frac{p - 16s^2 \mp \sqrt{f_p^-} - \sqrt{(p + 16s^2 \pm \sqrt{f_p^-})^2 - 4p^2}}{16s} \, . \tag{3.109}
$$

Then (3.105) is indeed the largest root provided also $c_p(s)$ or $d_p^{\pm}(s)$, respectively, is positive or complex.

We are thus reduced to determining the subset of $\{s > 0\}$ for which $f_p^{\pm}(s)$ is real and non-negative, and, respectively as appropriate, $a_p(s)$, $b_p^{\pm}(s)$ are strictly positive and $c_p(s)$, $d_p^{\pm}(s)$ are either strictly positive or complex. We refer to the two sign choices in $r_{\pm}$ as positive and negative branch solutions. The behaviour for $p = 1$ and $p = 2$ is again qualitatively different from that with $p \geq 3$.

### $p = 1$

#### Positive branch

The polynomial $f_1^+(s)$ is positive semidefinite for $s \in (0, \frac{\sqrt{2}-1}{4}] \cup [\frac{\sqrt{2}+1}{4}, \infty)$ but $a_1(s)$ is positive only for $s \in (0, \frac{\sqrt{2}-1}{4}]$. In this range $(1 - 16s^2 - \sqrt{f_1^+})^2 - 4$ is negative and so

$c_1(s)$ is complex; hence $r_+(Q_+(s))$ is indeed the largest root of $\Omega(r)$. In conclusion, for $s \in (0, \frac{\sqrt{2}-1}{4}]$ and $Q = Q_+(s)$ we have a regular 1/4 BPS solution on $\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$.

**Negative branches**

The polynomial $f_1^-(s)$ is positive semidefinite for $s \in (0, \frac{\sqrt{3}-\sqrt{2}}{4}] \cup [\frac{\sqrt{2}+\sqrt{3}}{4}, \infty)$ but $b_1^\pm(s)$ is positive only for $s \in (0, \frac{\sqrt{3}-\sqrt{2}}{4}]$. In this range $(1 + 16s^2 \pm \sqrt{f_1^-})^2 - 4$ is negative and so $d_1^\pm(s)$ is complex; hence $r_-(Q_-^\pm(s))$ is indeed the largest root of $\Omega(r)$. In conclusion, for $s \in (0, \frac{\sqrt{3}-\sqrt{2}}{4}]$ and $Q = Q_-^\pm(s)$ we have two regular 1/4 BPS solutions on $\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$.

### $p = 2$

**Positive branch**

For $p = 2$ the expressions for $r_+(Q_+(s))$ and $Q_+(s)$ simplify to

$$r_+(Q_+(s)) \;=\; \frac{1}{4s} - s \,, \qquad Q_+(s) \;=\; \frac{1}{16s^2} - \frac{1}{2} + 2s^2 \,. \tag{3.110}$$

The above values satisfy (3.100) for $s \in (0, \frac{1}{2\sqrt{2}})$. In this range $a_2(s)$ is positive while $c_2(s)$ is complex, *i.e.* $r_+(Q_+(s))$ is indeed the largest root of $\Omega(r)$. In conclusion, for $s \in (0, \frac{1}{2\sqrt{2}})$ and $Q = Q_+(s)$ we have a regular 1/4 BPS solution on $\mathcal{M}_2 = \mathcal{O}(-2) \to S^2$. In the limit $s = \frac{1}{2\sqrt{2}}$, the root $r_+(Q_+(s)) = s = \frac{1}{2\sqrt{2}}$ which corresponds to a Taub-NUT solution.

**Negative branches**

The polynomial $f_2^-(s)$ is positive semidefinite for $s \in (0, \frac{2-\sqrt{2}}{4}] \cup [\frac{\sqrt{2}+2}{4}, \infty)$ but $b_2^\pm(s)$ is positive only for $s \in (0, \frac{2-\sqrt{2}}{4}]$. In this range $(2 + 16s^2 \pm \sqrt{f_2^-})^2 - 16$ is negative and so $d_2^\pm(s)$ is complex. In conclusion, for $s \in (0, \frac{2-\sqrt{2}}{4}]$ and $Q = Q_-^\pm(s)$ we have two regular 1/4 BPS solutions on $\mathcal{M}_2 = \mathcal{O}(-2) \to S^2$.

### $p \geq 3$

**Positive branch**

The polynomial $f_p^+(s)$ is positive definite for all $s > 0$ since it has imaginary roots and $a_p(s)$ is also positive for all $s > 0$. In this range $c_p(s)$ is positive and hence $r_+(Q_+(s))$ is indeed the largest root of $\Omega(r)$. In conclusion, for $s > 0$ and $Q = Q_+(s)$ we have a regular 1/4 BPS solution on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$.

**Negative branches**

The polynomial $f_p^-(s)$ is positive semidefinite for $s \in (0, \frac{\sqrt{2+p}-\sqrt{2}}{4}] \cup [\frac{\sqrt{2}+\sqrt{2+p}}{4}, \infty)$ but $b_p^\pm(s)$ is positive only for $s \in (0, \frac{\sqrt{2+p}-\sqrt{2}}{4}]$. In this range $d_2^\pm(s)$ is positive and hence

$r_-(Q_-^{\pm}(s))$ is indeed the largest root of $\Omega(r)$. In conclusion, for $s \in (0, \frac{\sqrt{2+p}-\sqrt{2}}{4}]$ and $Q = Q_-^{\pm}(s)$ we have two regular 1/4 BPS solutions on $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$.

It is important to remark that these various branches of solutions really *are* distinct solutions. In particular, one should verify that the two negative branch solutions are not diffeomorphic. We have checked this is this case by comparing the value of the square of the Ricci tensor $R_{\mu\nu}R^{\mu\nu}$ evaluated on the bolt $S^2$ at $r = r_0$ (this may be defined in a coordinate-independent manner as the fixed point set of $U(1)_r$, generated by $\partial_\psi$). Indeed, one easily computes the general expression

$$R_{\mu\nu}R^{\mu\nu} \quad = \quad 36 + \frac{4(P^2 - Q^2)^2}{(r^2 - s^2)^4} \; . \tag{3.111}$$

It is a simple exercise to compute this at $r = r_0$ for the various cases, and check that the solutions we claim are distinct give distinct values of this curvature invariant on the bolt.

#### Gauge field and spinors

Let us now turn to analysing the global properties of the gauge field. After a suitable gauge transformation, the latter can be written *locally* as

$$A \quad = \quad \frac{s}{r^2 - s^2}\left[-2Qr - \frac{1}{2s}(r^2 + s^2)(4s^2 - 1)\right]\sigma_3 \; . \tag{3.112}$$

In particular, this gauge potential is *singular* on the $S^2$ at $r = r_0$, but is otherwise globally defined on the complement $\mathcal{M}_p \setminus S^2$ of the bolt. The field strength $F = \mathrm{d}A$ is easily verified to be a globally defined smooth two-form on $\mathcal{M}_p$, with non-trivial flux through the $S^2$ at $r = r_0$. Indeed, for $Q = Q_\pm(s)$ one computes the period through the $S^2$ at $r_0(s) = r_\pm(Q_\pm(s))$ (respectively) to be

$$\begin{aligned}\int_{S^2} \frac{F}{2\pi} \quad &= \quad -\frac{2s}{r_0(s)^2 - s^2}\left[-2Q_\pm(s) r_0(s) - \frac{1}{2}(r_0(s)^2 + s^2)(4s^2 - 1)\right] \\ &= \quad \pm\frac{p}{2} - 1 \; .\end{aligned} \tag{3.113}$$

Thus the positive/negative branch solutions have a gauge field flux $\pm\frac{p}{2} - 1$ through the bolt, respectively. Both branches then induce the *same* spinors and global gauge field at conformal infinity, for fixed $p$ and $s$. The factor of $-1$ in the quantization condition (3.113) is precisely the same as for the 1/4 BPS Quaternionic-Eguchi-Hanson solutions (3.57) in section 3.3.3, and its relation to having globally well-defined spinors on the conformal boundary, invariant under $\mathcal{L}_{\partial_\psi}$, is precisely the same as the discussion around equation (3.60).

We note that the Dirac spin$^c$ spinors are smooth sections of the following bundles:

$$\begin{cases} \pi^*\left[\mathcal{O}(p-2)\oplus\mathcal{O}(0)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(p)\right] & \text{positive branch} \\ \pi^*\left[\mathcal{O}(-2)\oplus\mathcal{O}(-p)\oplus\mathcal{O}(-p-2)\oplus\mathcal{O}(0)\right] & \text{negative branch} \end{cases} . \tag{3.114}$$

When $p$ is even the boundary gauge field $A^{(3)}$ is a connection on $\mathscr{L}^{\frac{p}{2}-1}$, while when $p$ is odd it is a connection on $\mathscr{L}^{-1}$. The three-dimensional boundary spinors are correspondingly sections of $\mathscr{S}_0\otimes\mathscr{L}^{-1}$ and $\mathscr{S}\otimes\mathscr{L}^{-1}$, respectively (see appendix E).

#### Special solutions

For $p<3$ the positive branches described in section 3.5.2 terminate at $s=\frac{\sqrt{2}-\sqrt{2-p}}{4}$ while for $p\geq 3$ the positive branch exists for all $s>0$, but there are special solutions at $s=\frac{1}{2}$ and $s=s_p$. The negative branches terminate at $\frac{\sqrt{p+2}-\sqrt{2}}{4}$ for all $p$. In this section we describe these various special and/or limiting solutions.

#### Positive branches

For $p=1$ the positive branch exists for $s\in(0,\frac{\sqrt{2}-1}{4}]$. As usual the $s=0$ limit is singular, but the terminating solution with $s=\frac{\sqrt{2}-1}{4}$ is a regular solution. At this value of $s$ we have $f_1^+(s)=0$, although we have not found an invariant geometric interpretation of this characterization of the solution. For $p=2$ the positive branch exists for $s\in(0,\frac{1}{2\sqrt{2}})$, but here the terminating solution in the limit $s\to\frac{1}{2\sqrt{2}}$ degenerates to the Taub-NUT-AdS/$\mathbb{Z}_2$ solution, which of course has an orbifold singularity. Thus for $p=2$ the positive branch joins onto the Taub-NUT-AdS/$\mathbb{Z}_2$ solutions. Notice that, in contrast to the 1/2 BPS case, here the limiting Taub-NUT-AdS/$\mathbb{Z}_2$ solution has zero torsion, since $\frac{p}{2}-1=0$ when $p=2$.

For $p\geq 3$ the positive branch exists for all $s>0$, but there are some notable special solutions on this branch. Firstly, $s=\frac{1}{2}$ leads to a round metric on $S^3/\mathbb{Z}_p$, and thus this solution is a "round Lens filling solution", as dubbed in section 3.4. However, while for the 1/2 BPS solutions the round Lens filling solutions were terminating solutions that joined together the positive and negative branches, here it appears as a special point on the positive branch. Of course, it is not a surprise to see the self-dual Quaternionic-Eguchi-Hanson solution arise from the special value $s=s_p=\frac{p}{4\sqrt{p-1}}$, and this is another special solution on the $p\geq 3$ 1/4 BPS positive branch.

#### Negative branches

The negative branches terminate at $s=\frac{\sqrt{p+2}-\sqrt{2}}{4}$ for all $p\geq 1$. At this value of $s$ we have $f_p^-(s)=0$, and in fact the two negative branches become identical at this point, and thus *join together*. Again, we have not found a geometrical characterization of the condition that $f_p^-(s) = 0$. Notice that for $p \geq 10$ we have $\tilde{s}_p \equiv (\sqrt{p+2} - \sqrt{2})/4 > 1/2$, and therefore there exist two additional round Lens filling solutions on the negative branches. These are distinct solutions, as follows by comparing the curvature invariant (3.111) on the bolt $S^2$.

### 3.5.3 Moduli space of solutions

We have summarized the even more intricate branch structure of the 1/4 BPS solutions in Figure 3.3. In general the conformal boundary has biaxially squashed $S^3/\mathbb{Z}_p$ metric (3.64), with squashing parameter $s > 0$, and boundary gauge field given by (3.96). The 1/4 BPS fillings of this boundary may then be summarized as follows:

- For $p = 1$, the boundary $S^3$ with arbitrary squashing parameter $s > 0$ always has the Taub-NUT-AdS solution as filling, but for $s \in (0, \frac{\sqrt{2}-1}{4}]$ there is also a smooth positive branch solution with topology $\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$, while for $s \in (0, \frac{\sqrt{3}-\sqrt{2}}{4}]$ there are *two* negative branch solutions (which are connected to each other) of the same topology. The Taub-NUT, positive, and negative branch solutions are disconnected from each other; this in fact had to be the case, as we shall see in the next section that they have different constant free energy. Notice that the $s = \frac{1}{2}$ AdS$_4$ metric sits on the Taub-NUT-AdS branch.

- For $p \geq 2$ and arbitrary squashing parameter $s > 0$ we always have the (mildly singular) Taub-NUT-AdS/$\mathbb{Z}_p$ solution. Thus for all boundary data there always exists a gravity filling, provided one allows for orbifold singularities.

- For $p = 2$ there is a positive branch filling for $s \in (0, \frac{1}{2\sqrt{2}})$ with topology $\mathcal{M}_2 = \mathcal{O}(-2) \to S^2$. This joins onto the Taub-NUT-AdS/$\mathbb{Z}_2$ branch at $s = \frac{1}{2\sqrt{2}}$, and we shall indeed see that these have the same free energy. Notice that, since $\frac{p}{2} - 1 = 0$ for $p = 2$, the gauge field is a connection on a trivial line bundle. For $s \in (0, \frac{2-\sqrt{2}}{4}]$ there are again two negative branch solutions. These are connected to each other, but disconnected from the positive branch and Taub-NUT-AdS branch.

- For all $p > 2$ and $s > 0$ there exists a positive branch filling with topology $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$. This includes the Quaternionic-Eguchi-Hanson solution at the specific value $s = s_p = \frac{p}{4\sqrt{p-1}}$, and the round Lens filling solution at $s = \frac{1}{2}$. However, this positive branch is disconnected from the Taub-NUT-AdS branch. For $s \in (0, \frac{\sqrt{p+2}-\sqrt{2}}{4}]$ there are again two negative branch solutions, which are connected to each other but disconnected from the positive branch and Taub-NUT-AdS branch. For $p \geq 10$ there exist two additional distinct round Lens filling solutions on the negative branches.

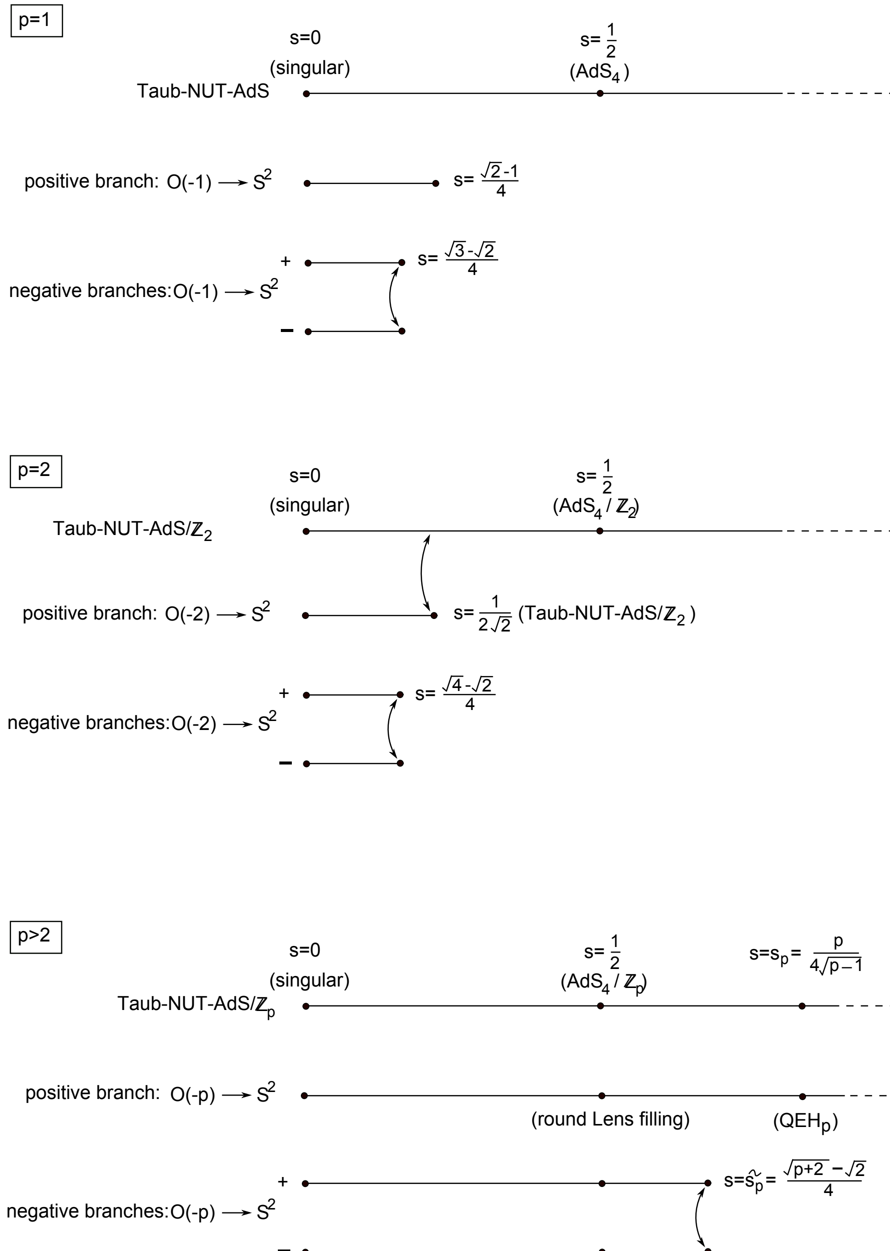


Figure 3.3: The moduli space of 1/4 BPS solutions with biaxially squashed $S^3/\mathbb{Z}_p$ boundary, with squashing parameter $s$. The arrows denote identification of solutions on different branches. Notice that these moduli spaces are generally disconnected, as follows from the fact that the free energies are different. Note also that the negative branches extend past the round Lens filling solutions at $s = \frac{1}{2}$ only when $p \geq 10$ (which is the case plotted).

### 3.5.4 Holographic free energy

In this subsection we compute the holographic free energy of the 1/4 BPS solutions summarized above. This follows similarly section 3.4.4, thus we will be more brief. Again we refer to appendices F.1 and F.2 for further details. We compute

$$I^{\text{grav}}_{\text{bulk}} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\left(2sr^3 - 6s^3 r\right)\Big|_{r_0}^{\varrho} , \tag{3.115}$$

$$I^{\text{grav}}_{\text{ct}} + I^{\text{grav}}_{\text{bdry}} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\left[4Qs^2 - 2s\varrho^3 + 6s^3\varrho + \mathcal{O}(\varrho^{-1})\right] , \tag{3.116}$$

where $r_0 = r_\pm$ is the appropriate largest root of $\Omega(r)$, where the manifold closes off. Removing the cut-off $\varrho \to \infty$ the divergent terms cancel. The contribution to the action from the bulk gauge field is as follows. For the NUT case $r_0 = s$ and we have

$$I^{F}_{\text{NUT}} = \frac{16\pi^2}{8\pi G_4}Q^2 = \frac{2\pi}{G_4}\frac{(1-4s^2)^2}{4} , \tag{3.117}$$

while for the Taub-Bolt-AdS cases $r_0 > s$ and we have

$$I^{F}_{\text{Bolt}} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}\frac{sr_0\left[-4Q(1-4s^2)(r_0+s)^2 + (r_0^2+s^2)(2Q+1-4s^2)^2\right]}{2(r_0^2-s^2)^2} . \tag{3.118}$$

Combining all the above contributions to the action we obtain the following remarkably simple expressions

$$\begin{aligned} I_{\text{NUT}} &= \frac{1}{2}\frac{\pi}{G_4} \qquad (p=1) , \\ I_{\text{Bolt}\pm} &= \frac{4\mp p}{8}\frac{\pi}{G_4} \qquad (p \geq 2) . \end{aligned} \tag{3.119}$$

Again, $I_{\text{Bolt}_\pm}$ refers to the free energies of the positive and negative branch solutions, respectively. In particular, the two distinct (non-diffeomorphic) negative branches in fact have the *same* free energy, that we denote $I_{\text{Bolt}_-}$.

As for the 1/2 BPS solutions, for any $p \geq 2$ we can fill the boundary squashed Lens space $S^3/\mathbb{Z}_p$ with the 1/4 BPS Taub-NUT-AdS/$\mathbb{Z}_p$ solution, where $\mathbb{Z}_p$ acts on the coordinate $\psi$. Here we must consider more specifically the orbifold NUT solutions with $\pm\frac{p}{2} - 1$ units of flux trapped at the orbifold singularity, as a direct quotient of the Taub-NUT-AdS solution is not supersymmetric. The latter solutions have the same global boundary data as the Taub-Bolt-AdS solutions, and in particular the trapped flux induces the same topological class of the gauge field on the conformal boundary $S^3/\mathbb{Z}_p$. Using the result of appendix F.2 we compute the total action

$$I^{\text{orbifold}}_{\text{NUT+flux}\pm} = \frac{1}{p}I_{\text{NUT}} + I_{\text{sing}} = \left(\frac{1}{2p} + \left(\frac{p}{2}\mp 1\right)^2\frac{1}{2p}\right)\frac{\pi}{G_4} , \tag{3.120}$$

where in this case we obtain two different values depending on the sign of the flux. In Figure 3.4 we plotted the holographic free energies for various values of $p$. The most striking feature is that we now have *four* distinct smooth supergravity solutions filling a squashed $S^3$ boundary ($p = 1$). The corresponding free energies are shown in the first plot.

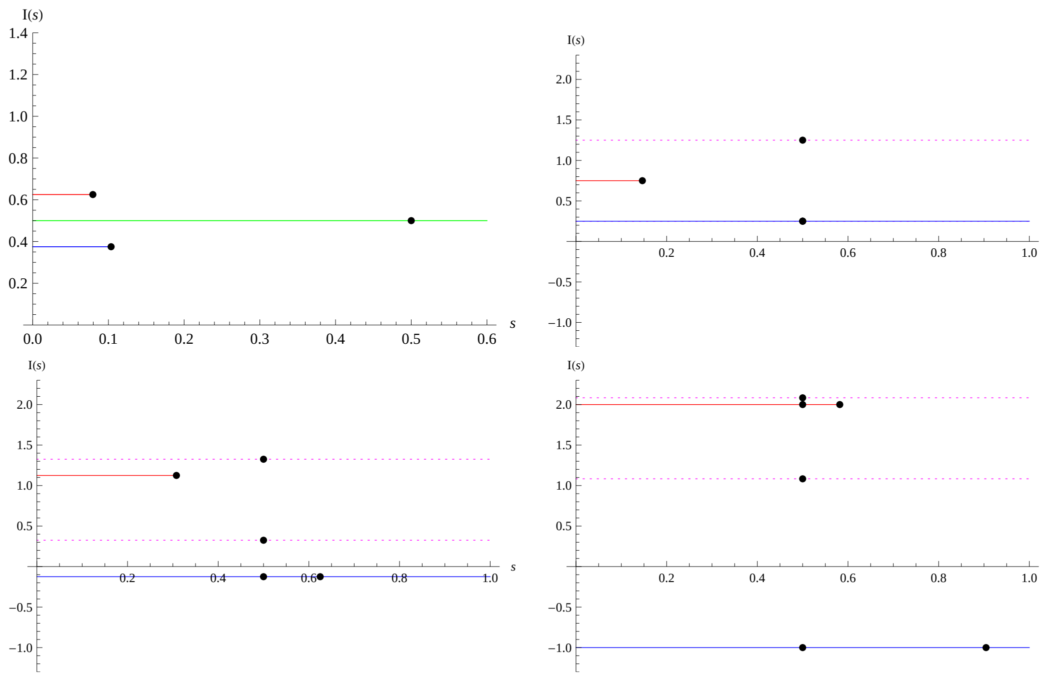


Figure 3.4: Plots of the free energies $I(s)$ of the different branches for $p = 1, 2, 5, 12$, respectively. The dotted lines in magenta are the free energies $I^{\text{orb}}_{\text{NUT+flux}_\pm}$, including the contribution of $\pm\frac{p}{2} - 1$ units of flux at the orbifold singularity. The red lines are the free energies $I_{\text{Bolt}_-}$ of the negative branches. The blue lines are the free energies $I_{\text{Bolt}_+}$ of the positive branches. The special solutions are marked with points.

## 3.6 M-theory solutions and holography

In this section we discuss how the four-dimensional supergravity solutions uplift to solutions of eleven-dimensional supergravity. The full eleven-dimensional solution will take the form of a fibration over $\mathcal{M}^{(4)}$, where the fibres are copies of the internal space $Y_7$. The choice of the latter determines the field theory dual that is defined on the biaxially squashed $S^3/\mathbb{Z}_p$ conformal boundary of $\mathcal{M}^{(4)}$. Recall that for all solutions the four-dimensional gauge field $A$ satisfies the quantization condition for a spin$^c$ gauge field, and in particular $2A$ is always a connection on a line bundle $L$ over $\mathcal{M}^{(4)}$. As we shall see, the Taub-NUT-AdS solutions may *always* be uplifted to global supersymmetric M-theory solutions, for any choice of internal space $Y_7$, and in this case we are able to compare the free energies computed in sections 3.4 and 3.5 to corresponding large $N$ field theory results, and find agreement in section 3.6.2. An important point here is that the Taub-NUT-AdS solutions have topology $\mathcal{M}^{(4)} \cong \mathbb{R}^4$, so that the line bundle $L$ is necessarily topologically trivial, *i.e.* the four-dimensional graviphoton $A$ is globally a one-form on $\mathcal{M}^{(4)}$. However, as soon as $c_1(L) \in H^2(\mathcal{M}^{(4)}, \mathbb{Z})$ is non-zero this puts constraints on the possible choices of $Y_7$ – this is the case for almost[15] all of the Taub-Bolt-AdS solutions,

[15]The exception is the 1/4 BPS positive branch solution with $p = 2$, which is the only case where $A$ is globally a one-form on $\mathcal{M}^{(4)}$. This then also uplifts for *any* choice of internal space $Y_7$. However, notice that

and even the Taub-NUT-AdS/$\mathbb{Z}_p$ solutions if they have non-trivial flat connections turned on.[16]

This may be rephrased as follows. Given any supersymmetric field theory with an $\mathrm{AdS}_4 \times Y_7$ gravity dual, this field theory may also be put on the biaxially squashed $S^3$, preserving 1/2 or 1/4 supersymmetry. Any such field theory then has a Taub-NUT-AdS filling as a gravity dual, of the form $\mathcal{M}^{(4)} \times Y_7$ where $\mathcal{M}^{(4)}$ is the Taub-NUT-AdS solution with appropriate 1/2 BPS or 1/4 BPS instanton, respectively. However, only a certain class of field theories, meaning only certain choices of $Y_7$, has in addition the 1/4 BPS Taub-Bolt-AdS filling of section 3.5. Similar comments apply to the case of the biaxially squashed Lens spaces $S^3/\mathbb{Z}_p$. We shall describe some choices of corresponding $Y_7$ in section 3.6.4, and comment on the dual field theories.

### 3.6.1 Lifting NUTs

As shown in [70], any supersymmetric solution to $d = 4$, $\mathcal{N} = 2$ gauged supergravity theory uplifts *locally* to a supersymmetric solution of $d = 11$ supergravity. More precisely, given any Sasaki-Einstein seven-manifold $Y_7$ with contact one-form $\eta$, transverse Kähler-Einstein metric $\mathrm{d}s^2_T$ and with the seven-dimensional metric normalized so that $R_{ij} = 6g_{ij}$, we have the uplifting ansatz[17]

$$\begin{aligned} \mathrm{d}s^2_{11} &= R^2\left(\frac{1}{4}\mathrm{d}s^2_4 + \left(\eta + \frac{1}{2}A\right)^2 + \mathrm{d}s^2_T\right) , \\ G &= R^3\left(\frac{3}{8}\mathrm{vol}_4 - \frac{1}{4} *_4 F \wedge \mathrm{d}\eta\right) . \end{aligned} \tag{3.121}$$

Here $\mathrm{d}s^2_4$ is the four-dimensional gauged supergravity metric on $\mathcal{M}^{(4)}$, with volume form $\mathrm{vol}_4$, and the radius $R$ is

$$R^6 = \frac{(2\pi\ell_p)^6 N}{6\mathrm{Vol}(Y_7)} , \tag{3.122}$$

where $N$ is the number of units of flux

$$N = \frac{1}{(2\pi\ell_p)^6}\int_{Y_7} *_{11} G . \tag{3.123}$$

the free energy (3.119) of this solution is equal to the free energy of $\mathrm{AdS}_4/\mathbb{Z}_2$, which has the same global boundary conditions.

[16]As we shall see, in general the uplifting to eleven-dimensions involves not $L$, but rather $L^{\lambda/2}$ for some rational $\lambda \in \mathbb{Q}$. Since $c_1(L\mid_{S^3/\mathbb{Z}_p}) \in \mathbb{Z}_p \cong H^2(S^3/\mathbb{Z}_p, \mathbb{Z})$ is always torsion when restricted to the boundary $S^3/\mathbb{Z}_p$, this will be crucial when we come to ask which solutions have the same *global* boundary conditions.

[17]A caveat here is that the uplifting formulae above were shown in [70] in Lorentzian signature. Passing to Euclidean signature does not affect this at the level of equations of motion. Global aspects of the eleven-dimensional Killing spinors are discussed in appendix E.3.

The four-dimensional Newton constant is then given by

$$\frac{1}{16\pi G_4} = N^{3/2}\sqrt{\frac{\pi^2}{32\cdot 27\mathrm{Vol}(Y_7)}} . \tag{3.124}$$

In fact it was more generally conjectured in [70] that given any $\mathcal{N}=2$ warped $\mathrm{AdS}_4\times Y_7$ solution of eleven-dimensional supergravity there is a consistent Kaluza-Klein truncation on $Y_7$ to $d=4$, $\mathcal{N}=2$ gauged supergravity theory. Properties of such general solutions were investigated in chapter 2, and we expect the contact structure to play an important role in this truncation. In particular, we saw that (3.124) remains true in this more general setting, provided one replaces the Riemannian volume $\mathrm{Vol}(Y_7)$ by the contact volume.

As a specific example we may consider simply $Y_7 = S^7/\mathbb{Z}_k$, with the $\mathbb{Z}_k$ action along the Hopf fibre of $S^7$. In this case $\mathrm{d}s^2_7$ is the usual Fubini-Study metric on $\mathbb{CP}^3$, and $\eta = \mathrm{d}\xi + A_{\mathbb{CP}^3}$, where $\xi$ has period $2\pi/k$ and $\mathrm{d}A_{\mathbb{CP}^3}$ is the Kähler form on $\mathbb{CP}^3$, normalized to have period $2\pi$ through the linearly embedded $\mathbb{CP}^1$. In that case $\mathrm{Vol}(S^7/\mathbb{Z}_k) = \pi^4/3k$. Different choices of $Y_7$ correspond to differerent choices of Chern-Simons-matter theory on the squashed $S^3$, and there are by now many examples of dual pairs, including infinite families.

The Taub-NUT-AdS solutions have topology $\mathcal{M}^{(4)} \cong \mathbb{R}^4$, and then necessarily $A$ is globally a one-form on $\mathbb{R}^4$. It follows immediately from the uplifting formula (3.121) that we obtain a globally supersymmetric eleven-dimensional solution, again of the product topology $\mathcal{M}^{(4)}\times Y_7$, for any choice of $\mathrm{AdS}_4\times Y_7$ solution. Specifically, because $A$ is a global one-form on $\mathcal{M}^{(4)}$, the twisting $\eta + \frac{1}{2}A$ is topologically trivial. Notice also that there is no flux quantization condition on $G$, since $\mathrm{d}\eta$ is exact. Thus any supersymmetric field theory on $S^3$ with an $\mathrm{AdS}_4\times Y_7$ dual also has, when the theory is put on the biaxially squashed $S^3$, a supersymmetric (Taub-NUT-AdS) $\times\, Y_7$ dual, in both the 1/2 BPS and 1/4 BPS cases.[18] We may then compare the gravitational holographic free energies of these solutions to corresponding exact large $N$ field theory computations, which we will do in the next section.

### 3.6.2 Comparison to field theory duals

The gravitational holographic free energies of the 1/2 BPS and 1/4 BPS Taub-NUT-AdS solutions were computed in sections 3.4.4 and 3.5.4, respectively. The result is

$$I_{\mathrm{NUT}} = \begin{cases} \dfrac{\pi}{2G_4} & 1/4\ \mathrm{BPS} \\ \dfrac{(2s)^2\pi}{2G_4} & 1/2\ \mathrm{BPS} \end{cases} . \tag{3.125}$$

[18] An interesting subtlety here is that when the squashing parameter $s$ satisfies $0<s<1/2$ the gauge field is in fact *complex*. One then formally obtains a complex eleven-dimensional metric via (3.121). This is the only case in which we obtain a non-real gauge field.

Using the formula (3.124) for the four-dimensional Newton constant, we thus obtain

$$I_{\mathrm{NUT}} = \begin{cases} \dfrac{\sqrt{2}\pi}{3}\sqrt{\dfrac{\mathrm{Vol}(S^7)}{\mathrm{Vol}(Y_7)}}N^{3/2} & 1/4 \text{ BPS} \\ (2s)^2\dfrac{\sqrt{2}\pi}{3}\sqrt{\dfrac{\mathrm{Vol}(S^7)}{\mathrm{Vol}(Y_7)}}N^{3/2} & 1/2 \text{ BPS} \end{cases} . \tag{3.126}$$

In fact the 1/2 BPS case was precisely studied in [81]. In this case the biaxially squashed $S^3$ with metric (3.26), boundary gauge field (3.27) and three-dimensional Killing spinor equation (3.25) was studied in [39]. In the latter reference the authors showed that, for a large class of $\mathcal{N} = 2$ Chern-Simons-quiver gauge theories, the leading large $N$ free energy is precisely $(2s)^2$ times the result for the round sphere (see equation (148) in [39]). This is precisely what we obtain from the 1/2 BPS Taub-NUT-AdS gravity solution (3.126), which has the same conformal boundary data!

In the 1/4 BPS case the boundary three-metric (3.26) is the same as in the 1/2 BPS case, but the boundary gauge field (3.36) and three-dimensional Killing spinor equation (3.35) are different. General $\mathcal{N} = 2$ Chern-Simons-matter theories were studied on this biaxially squashed $S^3$ in [38], and it was found that the partition function is *independent of the squashing parameter*. This is an exact statement, valid for all $N$. This then precisely agrees with our large $N$ gravity result in (3.126), where we find that the gravitational free energy is equal to the result for the round sphere with $s = \frac{1}{2}$. Thus the 1/4 BPS Taub-NUT-AdS solution reproduces the correct large $N$ free energy.[19] Of course, this can only be regarded as a partial result at this stage, because in the 1/4 BPS case there is also the Taub-Bolt-AdS filling, with topology $\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$. We turn to these solutions next.

### 3.6.3 Lifting bolts

The Taub-Bolt-AdS solutions certainly uplift *locally* to eleven-dimensional supersymmetric supergravity solutions via (3.121). However, globally this uplifting ansatz is inconsistent unless one restricts the internal space $Y_7$ appropriately. In this section we explain this important global subtlety. This implies that only a restricted class of field theories have Taub-Bolt-AdS fillings, in addition to the universal Taub-NUT-AdS fillings described in the previous section.

The discussion that follows is entirely topological, and we may in fact treat all of the 1/2 BPS and 1/4 BPS cases simultaneously. Specifically, all that we shall need to know is that the topology of the Taub-Bolt-AdS solutions is $\mathcal{M}^{(4)} = \mathcal{M}_p \equiv \mathcal{O}(-p) \to S^2$, with

[19] Notice that it is non-trivial that the final result is independent of the squashing parameter – each term in the action depends on $s$, with the $s$-dependence only canceling when all terms are summed.

the gauge field flux quantized as

$$\int_{S^2} \frac{F}{2\pi} = \frac{n}{2} . \tag{3.127}$$

In all cases $n \equiv p \bmod 2$, which is equivalent to $A$ be a spin$^c$ gauge field, as discussed in detail in appendix E.

For simplicity, we shall consider first the case of uplifting when the internal manifold $Y_7$ is a *regular* Sasaki-Einstein manifold. By definition this means that $Y_7$ is the total space of a $U(1)$ principal bundle over a Kähler-Einstein six-manifold $B_6$ with metric $\mathrm{d}s^2_T$. We may then write $\eta = \mathrm{d}\xi + \sigma$, where standard formulae give $\mathrm{d}\sigma = \rho/4$ where $\rho$ is the Ricci-form on $B_6$. The canonical period for $\xi$ is then $2\pi/4$, which for a Sasaki-Einstein manifold with precisely two Killing spinors is also the smallest period compatible with supersymmetry: the Killing spinors on $Y_7$ are charged under the Reeb vector $\partial_\xi$, and taking $\xi$ to have period $2\pi/4m$ for any $m > 1$ would lead to spinors that are not single-valued. When $\xi$ has period $2\pi/4$ $Y_7$ is in fact the total space of the $U(1)$ principal bundle associated to the anti-canonical line bundle over $B_6$. On the other hand, $\xi$ can sometimes have *larger* period. In fact $Y_7$ is simply-connected if and only if $\xi$ has period $2\pi I/4$ where $I = I(B_6) \in \mathbb{N}$ is a positive integer called the *Fano index* of $B_6$ [59]. In particular, for $B_6 = \mathbb{CP}^3$ we have $I(\mathbb{CP}^3) = 4$, so that for $Y_7 = S^7$ we must take $\xi$ to have period $2\pi$.[20] In fact $I \in \{1, 2, 3, 4\}$, with $I = 4$ only for $B_6 = \mathbb{CP}^3$.[21]

We may summarize the previous paragraph, then, by taking $\xi$ to have period $2\pi I/4k$, where $I = I(B_6) \in \{1, 2, 3, 4\}$ is the Fano index, and the positive integer $k$ must then divide $I$ in order that the two $U(1)_R$ charged Killing spinors are single-valued. The number of cases is then very small.

The global restriction on the internal space $Y_7$ in the uplifting ansatz (3.121) may then be understood by fixing a point in $B_6$ and looking at the corresponding circle bundle over the bolt $S^2 \subset \mathcal{M}^{(4)}$. Since $\xi$ has period $2\pi I/4k$, it follows from the connection term $\eta + \frac{1}{2}A$ appearing in the metric (3.121) that we will obtain a well-defined circle bundle only if

$$\frac{4k}{2I} \int_{S^2} \frac{F}{2\pi} = m \in \mathbb{Z} \tag{3.128}$$

is an integer. Geometrically, this integer $m$ is (minus) the first Chern class of the circle bundle, with coordinate $\xi$, integrated over the bolt $S^2$. Recalling that $2A$ is a connection on what we called $L \to \mathcal{M}^{(4)}$, we thus see that the eleven-dimensional circle $\xi$ is twisted by the line bundle[22] $L^{k/I} = \mathcal{O}(m)$ in general, rather than by $L$. When $k = I$ these are the

[20] In this case the discussion of Killing spinors is somewhat modified compared with that for a generic Sasaki-Einstein manifold: $S^7/\mathbb{Z}_k$ preserves $\mathcal{N} = 6$ supersymmetry for $k > 2$. The six Killing spinors here are invariant under the $\mathbb{Z}_k$ action for all $k \in \mathbb{Z}$.

[21] For completeness we note that examples exist for all values of $I \in \{1, 2, 3, 4\}$: $Y_7 = V^{5,2} = SO(5)/SO(3)$ has $I = I(\mathrm{Gr}(5, 2)) = 3$; $Y_7 = Q^{1,1,1,}$ has $I = I(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1) = 2$; $Y_7 = M^{3,2}$ has $I = I(\mathbb{CP}^1 \times \mathbb{CP}^2) = 1$.

[22] Notice that this means the rational number $\lambda$ we alluded to in footnotes 7 and 16 takes the value

same, which is precisely the case when the internal Sasaki-Einstein manifold $Y_7$ is the $U(1)$ principal bundle associated to the anti-canonical bundle over $B_6$. Given (3.127), the quantization condition (3.128) is equivalent to

$$nk \quad = \quad ml \ . \tag{3.129}$$

This necessary condition is then also sufficient for the eleven-dimensional metric (3.121) to be globally well-defined. Specifically, the eleven-dimensional spacetime is by construction the total space of the circle bundle over $\mathcal{M}_p \times B_6$ with first Chern class $c_1 = -m\Phi - \frac{k}{l}c_1(B_6)$, where $\Phi$ is the generator of $H^2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$. In the $G$-flux in (3.121) notice that now $\mathrm{d}\eta$ is no longer an exact form on the eleven-dimensional spacetime. In fact its cohomology class is equal to the cohomology class of $-\frac{1}{2}F$. But then $*_4F \wedge F$ is proportional to the volume form on $\mathcal{M}_p$, which is exact on $\mathcal{M}_p$ and thus also is exact on the eleven-dimensional spacetime. It follows that there is no quantization condition on $G$. In appendix E.3 we show that if the eleven-dimensional metric is regular then the eleven-dimensional geometry is always a spin manifold (for all $p$), and the eleven-dimensional Killing spinors are smooth and globally defined.

Taking $k = l$, which leads to the canonical period of $2\pi/4$ for $\xi$, we see that the condition (3.129) is always satisfied. Thus *all* Taub-Bolt-AdS solutions can be uplifted for *all* regular Sasaki-Einstein $Y_7$ with the canonical period of $2\pi/4$ for $\xi$. This is true for any $p$. Examples are then $Y_7 = S^7/\mathbb{Z}_4$, $Y_7 = V^{5,2}/\mathbb{Z}_3$, $Y_7 = Q^{2,2,2} = Q^{1,1,1}/\mathbb{Z}_2$, and $Y_7 = M^{3,2}$. In this case the Reeb $U(1)$ principal bundle, with fibre coordinate $\xi$, is twisted over the base spacetime $\mathcal{M}^{(4)}$ by the line bundle $L$.

However, more generally (3.129) leads to restrictions. Consider the case of $Y_7 = S^7$, which has $l = 4$ and $k = 1$. It follows from (3.129) that $n$ is necessarily divisible by 4. But recall that $n \equiv p \mod 2$, so we see immediately that *none* of the Taub-Bolt-AdS solutions with $p$ odd can be uplifted on the round seven-sphere! In particular, the 1/4 BPS Taub-Bolt-AdS solution that fills the squashed $S^3$ cannot be lifted on $S^7$ (nor can it be lifted on $S^7/\mathbb{Z}_2$, although from the previous paragraph it *can* be lifted on $S^7/\mathbb{Z}_4$). Concretely, this means that *the 1/4 BPS Taub-Bolt-AdS filling of the squashed $S^3$ does not exist for the ABJM theory*. We shall discuss this further in section 3.6.4 below. Other cases may be analysed similarly. For example, again taking $Y_7 = S^7$, which has $l = 4$ and $k = 1$, the 1/2 BPS solutions have $n = \pm p$, which leads to the restriction $p = \pm 4m$, so that the 1/2 BPS Taub-Bolt-AdS solutions uplift on $S^7$ only if $p$ is divisible by 4.

Above we have focused on regular Sasaki-Einstein manifolds $Y_7$, but it is straightforward to extend this analysis. *Irregular* Sasaki-Einstein manifolds have $\partial_\xi$ with generically non-closed orbits. This means that the coordinate $\xi$ is not periodically identified over a dense open subset of $Y_7$. On the other hand, the expression $\eta + \frac{1}{2}A$ defines a global one-form only if $\xi$ is periodically identified in $\eta = \mathrm{d}\xi + \sigma$. Thus one can *never*[23]

$\lambda = 2k/l$.

[23] However, see footnote 15.

lift any of these bolt solutions on irregular Sasaki-Einstein manifolds.

Finally, we conclude this section by commenting on an equivalent way of seeing the restriction on $Y_7$, that perhaps more directly makes contact with the field theory dual description. For simplicity, we again take $Y_7$ to be a regular Sasaki-Einstein manifold with Kähler-Einstein base $B_6$, Fano index $I = I(B_6)$ and $\xi$ to have period $2\pi I/4k$. It follows that $Y_7$ is the unit circle bundle in $\mathcal{L} = \mathcal{K}^{k/I}$, where $\mathcal{K}$ denotes the canonical line bundle of $B_6$. In this notation, scalar BPS operators arise in the dual field theory from *holomorphic functions* on the metric cone over $Y_7$. These correspond to holomorphic sections of $\mathcal{L}^{-t}$, with $t \in \mathbb{N}$ a positive integer. The R-charge of the holomorphic function is then proportional to $t$, namely $R = \lambda t = \frac{2k}{I} t$. However, because of the twisting in (3.121), these holomorphic functions become tensored with sections of a line bundle over the $S^2$ bolt. Specifically, in its dependence on $\mathcal{M}_p \times B_6$ a holomorphic function of R-charge $\frac{2k}{I} t$ becomes a section of $\mathcal{L}^{-t} \otimes \mathcal{O}(tm) \cong \mathcal{L}^{-t} \otimes L^{tk/I}$, where the integer $m$ satisfies (3.129). In the irregular case the holomorphic functions generically have irrational R-charges, which then do not lead to well-defined sections over the $S^2$ bolt.

### 3.6.4 Comments on field theory duals

We have seen that the 1/4 BPS Taub-Bolt-AdS filling of the biaxially squashed $S^3$ uplifts on any regular Sasaki-Einstein manifold with period $2\pi/4$ for $\xi$. Examples are $Y_7 = S^7/\mathbb{Z}_4$, $V^{5,2}/\mathbb{Z}_3$, $Q^{2,2,2}$ and $M^{3,2}$. Proposals for the corresponding field theory duals have been discussed in [18, 83, 84, 85, 86, 87, 88, 89]. However, the solution does *not* lift on the simply-connected covering spaces in the first three examples. We begin this section by examining this $p = 1$ case, noting that all other Taub-Bolt-AdS solutions fill the biaxially squashed Lens spaces $S^3/\mathbb{Z}_p$ with $p > 1$, and so far in the literature no one has studied $\mathcal{N} = 2$ supersymmetric gauge theories in this setting: the 1/2 BPS and 1/4 BPS biaxially squashed spheres were studied in [38], [39], and *round* Lens spaces $S^3/\mathbb{Z}_p$ *without torsion* gauge fields were studied in [82].

#### $S^3$ boundary

We first note that, thus far in the literature, the large $N$ limit of the partition function of the field theory models dual to $Q^{2,2,2}$ or $M^{3,2}$ has only been computed using an *ad hoc* prescription [90]. The issue is that the proposed field theory duals for these Sasaki-Einstein manifolds are *chiral*, meaning that the matter representation is not real, and this leads to a more complicated matrix model behaviour. In particular, it is possible that saddle points exist within these models corresponding to the Taub-Bolt-AdS solutions.

The $S^7/\mathbb{Z}_4$ case is also intriguing. Naively one might identify the field theory dual in this case with the ABJM model with $k = 4$; afterall, the ABJM theory is a $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory that is dual to the case $Y_7 = S^7/\mathbb{Z}_k$. However, the problem is quite subtle. The central issue is that the $\mathbb{Z}_k \subset U(1)$ quotient in the ABJM theory generally leaves $\mathcal{N} = 6$ supersymmetry unbroken, but the $U(1)$ R-symmetry that is being gauged when the theory is put on the squashed sphere corresponds to an $\mathcal{N} = 2$ subalgebra of this $\mathcal{N} = 6$. For the Taub-NUT-AdS solutions we may take $Y_7 = S^7/\mathbb{Z}_k$ and identify the $\xi$ circle in the uplifting ansatz (3.121) with a $U(1)_R \subset SO(6)$. Here the $\mathbb{Z}_k$ quotient is not contained in this $SO(6)$, where the latter rotates the $\mathcal{N} = 6$ supercharges in the vector representation. We are then gauging the manifest $U(1)_R$ symmetry of the ABJM when viewed in $\mathcal{N} = 2$ language. However, this does *not* work for the Taub-Bolt-AdS solutions on $\mathcal{M}_1 = \mathcal{O}(-1) \to S^2$, because we are forced to take $\xi$ to have period $2\pi/4$, *i.e.* the Taub-Bolt-AdS solutions are then defined with internal space $Y_7 = S^7/\mathbb{Z}_k \times \mathbb{Z}_4$. A dual field theory for the latter is then *unknown* (it is not simply an orbifold of the ABJM theory).

Of course, one might instead directly identify the $\mathbb{Z}_k$ quotient in the ABJM theory with with $\xi$ direction in the uplift (3.121). This then forces $k = 4$ for the Taub-Bolt-AdS solutions, and we are gauging a $U(1)_R$ symmetry that is *not* contained in the manifest $\mathcal{N} = 6$ supersymmetry of the ABJM theory with $k = 4$. This statement might puzzle some readers, since in the literature it is claimed that the ABJM theory has $\mathcal{N} = 6$ supersymmetry for all $k > 2$, while only $k = 1$ and $k = 2$ have enhanced $\mathcal{N} = 8$ supersymmetry. In fact this is incorrect, but subtly so. In fact there are 8 Killing spinors on $S^7/\mathbb{Z}_k$ for $k = 1, 2$ *and* $k = 4$, but for $k = 4$ the 2 additional Killing spinors are sections of a different spin bundle to the $\mathcal{N} = 6$ Killing spinors that exist on $S^7/\mathbb{Z}_k$ for all $k$. Recall that spin bundles on a manifold $\mathcal{M}$ are in general classified by $H^1(\mathcal{M}, \mathbb{Z}_2)$, and in the case at hand notice that $H^1(S^7/\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$. The $\mathcal{N} = 6$ spinors are sections of one of these two spin bundles, while the $\mathcal{N} = 2$ Killing spinors that exist when $S^7/\mathbb{Z}_4$ is viewed as a regular Sasaki-Einstein manifold over $\mathbb{CP}^3$ are sections of the other spin bundle.[24] Thus although there are 8 Killing spinors, there is *not* an $SO(8)$ R-symmetry that rotates them. In the field theory we are then gauging this non-manifest $\mathcal{N} = 2$ $U(1)_R$ symmetry that exists only when $k = 4$, which seems rather hard to study in practice.

The conclusion of this is that the internal spaces $Y_7$ for which the 1/4 BPS Taub-Bolt-AdS filling of $S^3$ exists do not currently have known field theory duals for which the large $N$ partition function computation is under good control: either the field theory models are chiral, and the large $N$ limit of the partition function is correspondingly not well-understood, or no field theory model is currently known, or the field theory *is* known and non-chiral, but we are gauging a classically non-manifest R-symmetry of that field theory.

### $S^3/\mathbb{Z}_p$ boundary

Let us now turn to the Lens space solutions for $p > 1$. Since in general there are a number of distinct cases to consider, we shall confine ourselves to commenting on what we believe are the more interesting cases/features.

[24]The corresponding situation for $S^3/\mathbb{Z}_p$ is discussed at length in appendix E.

Let us first discuss the solutions relevant for the ABJM model: in this case $Y_7 = S^7$ and correspondingly we have $l = 4$, $k = 1$, and hence $\lambda = \frac{1}{2}$. The latter is indeed the value of the R-charge of a chiral field in the ABJM field theory (these fields are usually called $A_1, A_2, B_1, B_2$), and the R-charges of gauge-invariant scalar chiral primary operators are $t/2$, where geometrically $t$ is the positive integer of section 3.6.3 (these operators are constructed using monopole operators of zero R-charge). Let us focus on the 1/2 BPS class of M-theory solutions. In this case, the Taub-Bolt-AdS solutions have *globally distinct* boundary conditions, as M-theory solutions, from the corresponding Taub-NUT-AdS/$\mathbb{Z}_p$ solution. To see this, note that from (3.129), and using $n = \pm p$, we see that a 1/2 BPS Taub-Bolt-AdS solution uplifts on $S^7$ only if $p = 4q$ is divisible by 4. In this case, $S^7$ is fibred over the base $\mathcal{M}_p$ by twisting the Hopf $S^1$ bundle by the line bundle $\mathcal{O}(m) = \mathcal{O}(\pm q)$. Alternatively, and equivalently, we may describe the total M-theory spacetime as the total space of the $U(1)$ principal bundle over $\mathcal{M}_p \times \mathbb{CP}^3$ with first Chern class $c_1 = \mp q\Phi - H$, where recall that $\Phi$ generates $H^2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$ and $H$ is the hyperplane class generating $H^2(\mathbb{CP}^3, \mathbb{Z}) \cong \mathbb{Z}$. However, since $\pm q \not\equiv 0 \mod p = 4q$, this $U(1)$ principal bundle is *also* non-trivially fibred over the boundary Lens space $S^3/\mathbb{Z}_p$. On the other hand, the Taub-NUT-AdS/$\mathbb{Z}_p$ solution is always trivially fibred.

To see what this means in terms of the dual boundary field theory, recall from the discussion at the end of section 3.6.3 that the functions on $S^7$ also become non-trivially fibred over $\mathcal{M}_p$ via the twisting, and in particular the Kaluza-Klein modes that are dual to the four chiral fields of the ABJM (or rather their gauge-invariants constructed using monopole operators), become sections of $\mathcal{O}(\pm q)$. This implies that for the Taub-Bolt-AdS solutions these basic matter fields are *twisted via their R-charge*, becoming sections of $\mathscr{L}^q$ rather than functions. We have attempted to study precisely this twisting in the 1/2 BPS case with $s = \frac{1}{2}$, since this is then (conjecturally) simply a twisted version of matrix model studied in [82]. It is straightforward to see that this twisting does indeed preserve supersymmetry, and that localization goes through similarly to the untwisted case. Our results so far are somewhat inconclusive: the behaviour of the matrix model is now much more involved, although interestingly we find that the Wilson loop VEV, discussed in appendix G, is indeed exactly zero, thus agreeing with the gravity prediction. We also find $N^{3/2}$ scaling of the free energy at large $N$, but with a coefficient that doesn't seem to match the gravity prediction of section 3.4.4. However, a key issue that affects both this example, and indeed all of the Taub-Bolt-AdS examples, is whether the (potential) twisting of the matter fields by their R-charge is the *only* effect on the Lagrangian of the untwisted theory, or whether the correct dual field theory is a more complicated deformation. For now we leave this issue open.

Having discussed an example where $l \neq k$, let us conclude this section with the class of Taub-Bolt-AdS solutions where the Sasaki-Einstein manifold has $k = l$, which then all uplift to M-theory. In this case notice that the circle bundle $\xi$ is twisted over the base $\mathcal{M}^{(4)}$ by the line bundle $L$. For the 1/2 BPS solutions this has first Chern class $\pm p$

through the bolt, implying that $L$ restricted to the boundary $S^3/\mathbb{Z}_p$ is *always trivial* in this 1/2 BPS class. This implies that all the 1/2 BPS solutions filling a fixed squashed $S^3/\mathbb{Z}_p$ in fact have the same global boundary data. In turn, in the dual field theory we then don't have the twisting by the flat R-symmetry Wilson line, discussed in the previous paragraph for the ABJM case. If the field theory Lagrangians are exactly the same in all cases, one should then compare the free energies of *all* the solutions plotted in Figure 3.2. However, to our knowledge all field theories within this class are *chiral models*, for which the matrix model is not under good control.

# 4 Conclusions

In this thesis we have explored supersymmetric backgrounds of M-theory in the context of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

We have determined the necessary and sufficient conditions on supersymmetric solutions of $d = 11$ supergravity that are dual to $\mathcal{N} = 2$ three-dimensional superconformal field theories. The eleven-dimensional metric is taken to be a warped product of $\mathrm{AdS}_4$ with a seven-dimensional Riemannian metric, and we have allowed for the most general four-form $G$ consistent with $SO(3,2)$ symmetry. We showed that generically the supersymmetry conditions may be formulated in terms of a canonical local $SU(2)$-structure on the seven-dimensional manifold $Y_7$. The well-known Freund-Rubin $\mathrm{AdS}_4 \times Y_7$ solutions where $Y_7$ is Sasaki-Einstein arise as a special case, characterized by an $SU(3)$-structure. For solutions with non-zero M2-brane charge, we showed that many geometrical and physical properties of $Y_7$ are captured by a contact structure. We also recovered the class of general solutions with vanishing M2-brane charge, previously discussed in [42].

By imposing a single additional requirement, that a certain vector bilinear is a Killing vector, we reduced the conditions to solving a second order non-linear ODE. The seven-dimensional metric on $Y_7$ is then fully specified by the choice of a (local) four-dimensional Kähler-Einstein metric, and any solution to this ODE. We managed to find an analytic solution of the ODE, and showed that this reproduces a class of solutions found originally in [27]. In addition, using a combination of analytic and numerical methods, we have discovered a further solution to our ODE, yielding a class of new supersymmetric $\mathrm{AdS}_4$ solutions with non-trivial four-form flux. These can be interpreted as holographically dual to certain cubic superpotential deformations of $\mathcal{N} = 2$ Chern-Simons gauge theories. When the Kähler-Einstein metric is chosen to be that on $\mathbb{CP}^2$, the seven-dimensional metric is a smooth (non-Einstein) metric on $S^7$, different from that of [27].

Our work may be regarded as providing the foundation for studying more general aspects of $\mathcal{N} = 2$ three-dimensional superconformal field theories with M-theory duals. For example, we expect that the geometric characterization of solutions we presented may be used to attack general problems, such as the gravity dual of $\mathcal{F}$-maximization, similarly to the developments in [45, 52]. It is also clear that using our results it will be possible to construct a consistent Kaluza-Klein truncation to four dimensions, extending that in [70]. Of course, it would also be very interesting to use our general equations as a method for finding new solutions (perhaps numerically), outside the classes that have been discovered so far.

We have presented all supersymmetric asymptotically locally $AdS_4$ solutions of Euclidean Einstein-Maxwell theory, possessing $SU(2) \times U(1)$ symmetry. We have shown that in general these solutions have one modulus, which is the squashing parameter $s$ of the Lens space metric at conformal infinity. However, we have also uncovered an intricate moduli space of solutions, comprising different branches, joining at special values of the parameter. Perhaps surprisingly, we found that typically for fixed conformal boundary data there exist multiple solutions, with different topologies. We studied global aspects of these solutions, finding a subtle interplay between bulk and boundary spin structures,

We showed that the Taub-Bolt-AdS solutions, despite being perfectly smooth and globally well-defined in four dimensions, can be uplifted to eleven-dimensional supergravity only for particular internal Sasaki-Einstein manifolds. Moreover, we showed that in these solutions the gauge field in the bulk induces non-zero gauge field on the boundary, whose global properties are intimately related to the specific Sasaki-Einstein manifold in the eleven-dimensional solution. Therefore, generically, the supersymmetric Taub-NUT-AdS solutions (and their orbifolds) are the only supersymmetric solutions filling a biaxially squashed Lens space. In particular, there exist only two distinct choices of instantonic gauge field such that the solutions preserve 1/2 or 1/4 supersymmetry, respectively. We have argued that these correspond to the two different constructions of supersymmetric field theories on a biaxially squashed three-sphere discussed in [38] and [39], respectively.

Nevertheless, there exist many examples where the Taub-Bolt-AdS solutions exist as global, smooth supersymmetric solutions of eleven-dimensional supergravity. In particular, we have shown that there exist (infinitely many) examples where fixed boundary data can be filled, supersymmetrically, with bulk solutions with different topologies, and with different holographic free energies. In order to address the problem of holographic dual field theories systematically, an important problem that remains open is the possible existence of *further* M-theory solutions, with the same boundary data as those we have found, but with smaller gravitational free energies. At present we can't exclude that such solutions exist outside the ansatz that leads to minimal gauged supergravity.

# A | Identities

In this appendix we collect a number of useful identities that have been used repeatedly to derive the results presented in chapter 2.

From the algebraic equation in (2.8) one can derive the following useful identities

$$\left(\bar{\chi}_i^c \mathcal{C}\chi_j^c + \bar{\chi}_i \mathcal{C}\chi_j\right) - \frac{\mathrm{i}m}{3}\mathrm{e}^{-3\Delta}\bar{\chi}_i^c \mathcal{C}\chi_j + \frac{1}{2}\bar{\chi}_i^c[\mathcal{C}, \partial\!\!\!/\Delta]_-\chi_j + \frac{1}{288}\mathrm{e}^{-3\Delta}\bar{\chi}_i^c[\mathcal{C}, F\!\!\!/]_+\chi_j = 0 \quad \text{(A.1)}$$

$$\left(\bar{\chi}_i^c \mathcal{C}\chi_j^c - \bar{\chi}_i \mathcal{C}\chi_j\right) + \frac{1}{2}\bar{\chi}_i^c[\mathcal{C}, \partial\!\!\!/\Delta]_+\chi_j + \frac{1}{288}\mathrm{e}^{-3\Delta}\bar{\chi}_i^c[\mathcal{C}, F\!\!\!/]_-\chi_j = 0 \quad \text{(A.2)}$$

where $\mathcal{C} \in \text{Cliff}(7)$ is an arbitrary element of the Clifford algebra and $[\ ,\ ]_\pm$ denotes the (anti)-commutator. Similarly we note

$$\left(\bar{\chi}_i^c \mathcal{C}\chi_j - \bar{\chi}_i \mathcal{C}\chi_j^c\right) + \frac{\mathrm{i}m}{3}\mathrm{e}^{-3\Delta}\bar{\chi}_i \mathcal{C}\chi_j - \frac{1}{2}\bar{\chi}_i[\mathcal{C}, \partial\!\!\!/\Delta]_-\chi_j - \frac{1}{288}\mathrm{e}^{-3\Delta}\bar{\chi}_i[\mathcal{C}, F\!\!\!/]_-\chi_j = 0 \quad \text{(A.3)}$$

$$\left(\bar{\chi}_i^c \mathcal{C}\chi_j + \bar{\chi}_i \mathcal{C}\chi_j^c\right) + \frac{1}{2}\bar{\chi}_i[\mathcal{C}, \partial\!\!\!/\Delta]_+\chi_j + \frac{1}{288}\mathrm{e}^{-3\Delta}\bar{\chi}_i[\mathcal{C}, F\!\!\!/]_+\chi_j = 0\ . \quad \text{(A.4)}$$

Similar identities exist in the alternative basis (2.12).

From the Fierz identity for the Cliff(7, 0) algebra

$$\begin{aligned} \bar{\chi}_1\chi_2\,\bar{\chi}_3\chi_4 &= \tfrac{1}{8}[\bar{\chi}_1\chi_4\,\bar{\chi}_3\chi_2 + \bar{\chi}_1\gamma_m\chi_4\,\bar{\chi}_3\gamma^m\chi_2 \\ &- \tfrac{1}{2!}\bar{\chi}_1\gamma_{mn}\chi_4\,\bar{\chi}_3\gamma^{mn}\chi_2 - \tfrac{1}{3!}\bar{\chi}_1\gamma_{mnp}\chi_4\,\bar{\chi}_3\gamma^{mnp}\chi_2]\ , \end{aligned} \quad \text{(A.5)}$$

where $\chi_a$, $a = 1, 2, 3, 4$, are arbitrary Spin(7) spinors, we derive the identity

$$\bar{\chi}_1^c\gamma^m\chi_2\,\bar{\chi}_2^c\gamma_m\chi_4 = \bar{\chi}_1^c\chi_4\,\bar{\chi}_2^c\chi_2 - \bar{\chi}_1^c\chi_2\,\bar{\chi}_2^c\chi_4\ . \quad \text{(A.6)}$$

# B Spinor bilinears

Recall that the $SU(2)$-structure is specified by two spinors $\chi_1$, $\chi_2$, or equivalently the linear combinations $\chi_\pm \equiv \frac{1}{\sqrt{2}}(\chi_1 \pm \mathrm{i}\chi_2)$ defined in (2.12). Here we choose to use $\chi_\pm$ as our basis. We then have the following bilinears:

| | |
|---|---|
| $\bar{\chi}_+\chi_+ = \bar{\chi}_-\chi_-$ | $1$ |
| $\bar{\chi}_+\chi_-$ | $0$ |
| $\mathrm{i}\bar{\chi}_+^c\chi_-$ | $\frac{m}{6}\mathrm{e}^{-3\Delta}$ |
| $\mathrm{i}\bar{\chi}_+^c\gamma_{(1)}\chi_-$ | $\xi e^7$ |
| $\bar{\chi}_-\gamma_{(1)}\chi_+$ | $\mathrm{i}\frac{S}{\xi}e^7 + \frac{S\sqrt{1-\xi^2}}{\|S\|}\left(e^5 - \mathrm{i}\frac{m}{6}\mathrm{e}^{-3\Delta}\frac{1}{\xi}e^6\right)$ |
| $\bar{\chi}_-\gamma_{(1)}\chi_- = -\bar{\chi}_+\gamma_{(1)}\chi_+$ | $\frac{m}{6}\mathrm{e}^{-3\Delta}\frac{1}{\xi}e^7 + \frac{\|S\|\sqrt{1-\xi^2}}{\xi}e^6$ |
| $\frac{1}{2\mathrm{i}}\left(\bar{\chi}_+\gamma_{(2)}\chi_+ + \bar{\chi}_-\gamma_{(2)}\chi_-\right)$ | $\sqrt{1-\xi^2}J_2$ |
| $\frac{1}{2\mathrm{i}}\left(\bar{\chi}_+\gamma_{(2)}\chi_+ - \bar{\chi}_-\gamma_{(2)}\chi_-\right)$ | $\frac{m}{6}\mathrm{e}^{-3\Delta}J_3 + \frac{1}{\xi}e^5 \wedge \left(\|S\|\sqrt{1-\xi^2}e^7 - \frac{m}{6}\mathrm{e}^{-3\Delta}e^6\right)$ |
| $\bar{\chi}_+^c\gamma_{(2)}\chi_-$ | $-J_3 + \xi e^{56} - \mathrm{i}\sqrt{1-\xi^2}J_1$ |

Table B.1: Spinor bilinears

# C | Solving the Einstein-Maxwell equations

In this section we find the general solution to Einstein-Maxwell equations (3.2) with $SU(2) \times U(1)$ symmetry. The ansatz for the metric and gauge field takes the form

$$\begin{aligned} \mathrm{d}s_4^2 &= \alpha^2(r)\mathrm{d}r^2 + \beta^2(r)(\sigma_1^2 + \sigma_2^2) + \gamma^2(r)\sigma_3^2 \ , \\ A &= h(r)\sigma_3 \ , \end{aligned} \tag{C.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are left-invariant one-forms for $SU(2)$, given explicitly by (3.5). In the following analysis we will use the local orthonormal frame

$$\begin{aligned} \hat{\mathrm{e}}^1 &= \beta(r)\mathrm{d}\theta \ , & \hat{\mathrm{e}}^2 &= \beta(r)\sin\theta\mathrm{d}\phi \ , \\ \hat{\mathrm{e}}^3 &= \gamma(r)(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi) \ , & \hat{\mathrm{e}}^4 &= \alpha(r)\mathrm{d}r \ , \end{aligned} \tag{C.2}$$

and introduce frame indices $a, b, c = 1, 2, 3, 4$. The Einstein equations read (with $\ell = 1$)

$$R_{ab} = -3\delta_{ab} + 2T_{ab} \ , \tag{C.3}$$

where $T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}F^2\,\delta_{ab}$ is the stress-energy tensor of the gauge field. For the ansatz (C.1) we compute

$$\begin{aligned} R_{44} &= -\frac{\gamma''}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma} - \frac{2\beta''}{\alpha^2\beta} + \frac{2\alpha'\beta'}{\alpha^3\beta} \ , \\ R_{33} &= -\frac{\gamma''}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma} - \frac{2\beta'\gamma'}{\alpha^2\beta\gamma} + \frac{\gamma^2}{2\beta^4} \ , \\ R_{11} = R_{22} &= -\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta} - \frac{\beta'\gamma'}{\alpha^2\beta\gamma} - \frac{\beta'^2}{\alpha^2\beta^2} + \frac{1}{\beta^2} - \frac{\gamma^2}{2\beta^4} \ , \\ T_{11} = T_{22} = -T_{33} = -T_{44} &= \frac{1}{2}\frac{h^2}{\beta^4} - \frac{1}{2}\frac{h'^2}{\alpha^2\gamma^2} \ , \end{aligned} \tag{C.4}$$

where a prime denotes derivative with respect to $r$. Furthermore, the equation of motion of the gauge field $\mathrm{d} * F = 0$ becomes

$$-\left(\frac{\beta^2}{\alpha\gamma}h'\right)' + \frac{\alpha\gamma}{\beta^2}h = 0 \ . \tag{C.5}$$

By considering the difference $R_{44} - R_{33}$ we obtain the equation

$$-\frac{2\beta''}{\alpha^2\beta} + \frac{2\beta'}{\alpha^2\beta}\left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right) - \frac{\gamma^2}{2\beta^4} \quad = \quad 0 \ , \tag{C.6}$$

and by an appropriate reparametrization of $r$ we can take

$$\alpha\gamma \quad = \quad 2s \ , \qquad \beta^2 \ = \ r^2 - s^2 \ . \tag{C.7}$$

The equation of motion for the gauge field then becomes an ordinary differential equation for $h(r)$:

$$-((r^2 - s^2)h')' + \frac{4s^2}{r^2 - s^2}h \quad = \quad 0 \ . \tag{C.8}$$

The general solution to (C.8) is easily found to be

$$h(r) \quad = \quad P\frac{r^2 + s^2}{r^2 - s^2} - Q\frac{2rs}{r^2 - s^2} \ , \tag{C.9}$$

where $P$ and $Q$ are integration constants. Substituting this back into the 33-component of the Einstein equation gives a second order ODE for the metric function $\gamma(r)$. The general solution to this is

$$\gamma^2(r) \quad = \quad \frac{4s^2}{r^2 - s^2}\left[P^2 - Q^2 - 2Mr + r^2(r^2 - 3s^2) + C\left(1 + \frac{r^2}{s^2}\right)\right] \ , \tag{C.10}$$

where $C$ and $M$ are two new integration constants. Substituting this into the 11-component of the Einstein equations then constrains

$$C \quad = \quad s^2(1 - 3s^2) \ . \tag{C.11}$$

This is precisely an analytic continuation the Reissner-Nordström-Taub-NUT-AdS (RN-TN-AdS) solution. Hence we have proven that this is the most general solution to the Einstein-Maxwell equations with $SU(2) \times U(1)$ symmetry.

# D Integrability conditions of Killing spinor equations

## D.1 BPS equations

In this appendix we compute the general integrability conditions for supersymmetry for the Euclidean RN-TN-AdS solutions derived in appendix C. An analysis for Lorentzian solutions was performed in [76].

The Euclidean RN-TN-AdS solutions are given by (3.6), (3.7). In this section we use the orthonormal frame $e^a$ in (3.14), which we note is *different* to the orthonormal frame $\hat{e}^a$ used in appendix C, and take the basis of gamma matrices (3.16). The integrability condition for the Killing spinor equation (3.3) reads[1]

$$\mathcal{I}_{ab}\,\epsilon \;=\; 0\;, \tag{D.1}$$

where

$$\begin{aligned}\mathcal{I}_{ab} \;\equiv\; & \tfrac{1}{4}R_{ab}{}^{cd}\Gamma_{cd} + \tfrac{1}{2}\Gamma_{ab} - \mathrm{i}F_{ab}\mathbb{I}_4 + \tfrac{\mathrm{i}}{2}\nabla_{[a}F_{|cd}\Gamma^{cd}{}_{|}\Gamma_{b]} + \tfrac{\mathrm{i}}{4}\Gamma_{[a}F_{|cd}\Gamma^{cd}{}_{|}\Gamma_{a]} \\ & - \tfrac{1}{16}\Big[F_{cd}\Gamma^{cd}\Gamma_a, F_{cd}\Gamma^{cd}\Gamma_b\Big] + \tfrac{\mathrm{i}}{4}F_{cd}\Gamma^{cd}\Gamma_{ab}\;,\end{aligned} \tag{D.2}$$

is a two-form with values in the Clifford algebra.

A necessary condition to have a non-trivial solution to (D.1) is that

$$\det{}_{\mathrm{Cliff}}\,\mathcal{I}_{ab} \;=\; 0\;, \tag{D.3}$$

holds for all $a, b$. We compute

$$\det{}_{\mathrm{Cliff}}\,\mathcal{I}_{ab} \;=\; \frac{-B_+B_- + D(B_+ - B_-)r + D^2r^2}{(r^2-s^2)^6}W_{ab}\;, \tag{D.4}$$

where

$$\begin{aligned} D \;&\equiv\; 2\Big[MP - sQ(1-4s^2)\Big]\;,\\ B_\pm \;&\equiv\; (M\pm sQ)^2 - s^2(1\pm P - 4s^2)^2 - (1\pm 2P - 5s^2)(P^2 - Q^2)\;, \end{aligned} \tag{D.5}$$

[1] We use frame indices $a, b, c, \ldots$.

and

$$(W_{ab}) \equiv \begin{pmatrix} 0 & 1 & \frac{1}{16} & \frac{1}{16} \\ 1 & 0 & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & 0 & 1 \\ \frac{1}{16} & \frac{1}{16} & 1 & 0 \end{pmatrix} . \tag{D.6}$$

We thus conclude that a *necessary* condition to have a supersymmetric solution is that the numerator in (D.4) is zero, which is equivalent to

$$D = 0 , \qquad B_+ B_- = 0 . \tag{D.7}$$

These can also be obtained from an analytic continuation of the integrability conditions in [76], but here we have derived the equations from first principles. We study the general solutions to (D.7) in section 3.2.2.

## D.2 Class III and supersymmetry

In this appendix we show that the condition $P = \pm Q$ characterizing Class III is not sufficient for supersymmetry, but rather the existence of a Killing spinor requires in addition

$$\begin{aligned} P &= -\frac{1}{2}(4s^2-1) , \quad \text{or} \\ P &= -s\sqrt{4s^2-1} . \end{aligned} \tag{D.8}$$

In order to prove this we look at the boundary Killing spinor equation, which can be derived from (3.3) upon expanding in powers of $1/r$. At lowest order we find

$$(\nabla^{(3)}_\alpha - \mathrm{i}A^{(3)}_\alpha)\chi - \frac{\mathrm{i}s}{2}\gamma_\alpha\chi + \mathrm{i}V_\beta\gamma_\alpha\gamma^\beta\chi = 0 . \tag{D.9}$$

Here $\nabla^{(3)}$ denotes the spin connection for the three-metric

$$\mathrm{d}s_3^2 = \sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2 , \tag{D.10}$$

with $\gamma_\alpha$, $\alpha = 1,2,3$ generating the corresponding Cliff(3,0) algebra, and $\chi$ is a two-component spinor. Furthermore

$$\begin{aligned} A^{(3)} &= \lim_{r\to\infty} A = P\sigma_3 , \\ V &= \frac{s^2(4s^2-1)}{Q}\sigma_3 . \end{aligned} \tag{D.11}$$

The integrability condition for (D.9) reads

$$\mathcal{I}^{(3)}_{\alpha\beta}\chi = 0 \,, \tag{D.12}$$

where

$$\begin{aligned}\mathcal{I}^{(3)}_{\alpha\beta} \equiv\ & \frac{1}{4}R^{(3)\,\alpha_1\alpha_2}_{\alpha\beta}\gamma_{\alpha_1\alpha_2} - \mathrm{i}F^{(3)}_{\alpha\beta} - \frac{s^2}{2}\gamma_{\alpha\beta} - 2\mathrm{i}\nabla_{[\alpha|}V_{\alpha_1}\gamma_{|\beta]}\gamma^{\alpha_1} \\ & - 2s\gamma_{[\alpha}V_{\beta]} + 2V^{\alpha_1}V_{\alpha_1}\gamma_{\alpha\beta} - 4V_{\alpha_1}\gamma_{[\alpha}V_{\beta]}\gamma^{\alpha_1} \ .\end{aligned} \tag{D.13}$$

A necessary condition to have a non-trivial solution to (D.12) is that

$$\det{}_{\mathrm{Cliff}}\,\mathcal{I}^{(3)}_{\alpha\beta} = 0 \ . \tag{D.14}$$

Taking into account $P = \pm Q$ we find that this is equivalent to

$$\frac{[(1-4s^2)^2 - 4Q^2][Q^2 + s^2(1-4s^2)]^2}{4Q^4} = 0 \,, \tag{D.15}$$

and hence (D.8) must hold.

# E | Spin$^c$ structures on bolt solutions

In this appendix we discuss in detail the spin$^c$ structures, in the bulk and on the conformal boundary, for the bolt-type solutions. This is a little subtle, because for $p$ odd the bolt solutions are not spin manifolds (but nevertheless are supersymmetric and admit Killing spinors). Correlated with this, the four-dimensional graviphoton in the bulk is in general a spin$^c$ connection, meaning that when $p$ is odd it is not a gauge field in the usual sense. We begin in section E.1 with a general topological discussion, and then in section E.2 give some more explicit details in the cases of interest. Section E.3 contains a brief discussion of lifting these spinors to eleven dimensions.

## E.1 Topological discussion

In general, recall that on an orientable four-manifold $\mathcal{M}^{(4)}$ the spin bundle $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ exists if and only if the second Stiefel-Whitney class is zero, so $w_2(\mathcal{M}^{(4)}) = 0 \in H^2(\mathcal{M}^{(4)}, \mathbb{Z}_2)$. However, it is also true that on *every* four-manifold the spin$^c$ bundles $\mathcal{S}_\pm \otimes L^{1/2}$ exist, where $L$ is a line bundle satisfying

$$c_1(L) \equiv w_2(\mathcal{M}^{(4)}) \mod 2 . \tag{E.1}$$

A spin$^c$ gauge field then has the property that $2A$ is a connection on $L$, so that (formally) $A$ is a connection on $L^{1/2}$.

Recall that the bolt-type solutions all have the topology $\mathcal{M}^{(4)} = \mathcal{M}_p$ = total space of $\mathcal{O}(-p) \to S^2$. A simple computation shows that $w_2(\mathcal{M}_p)$ is zero for $p$ even, while for $p$ odd $w_2(\mathcal{M}_p)$ generates the cohomology group $H^2(\mathcal{M}_p, \mathbb{Z}_2) \cong \mathbb{Z}_2$. We assume that the gauge field has field strength $F$ satisfying

$$\int_{S^2} \frac{F}{2\pi} = \frac{n}{2} , \tag{E.2}$$

where $S^2 \subset \mathcal{M}_p$ denotes the bolt/zero-section, so that $c_1(L) = n \in H^2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$. Then via (E.1), we see that $A$ is a spin$^c$ gauge field if and only if $n \equiv p \mod 2$. Notice that for all the solutions discussed in the main text regularity of the metric fixes the gauge field, and that $n \equiv p \mod 2$ was then indeed found to hold automatically for this gauge field. This is a necessary condition for supersymmetry.

In this section we would like to describe the spin$^c$ bundles $\mathcal{S}_\pm \otimes L^{1/2}$ more explicitly. We begin by noting that, although the metrics on $\mathcal{M}_p$ are not Kähler, nevertheless $\mathcal{M}_p$ admits a Kähler structure. We may then use the fact that on a Kähler four-manifold the spin bundles are (formally)

$$\begin{aligned} \mathcal{S}_+ &= K^{1/2} \oplus K^{-1/2} , \\ \mathcal{S}_- &= K^{1/2} \otimes \Omega^{0,1} . \end{aligned} \tag{E.3}$$

Here $K$ denotes the canonical line bundle, while $\Omega^{0,1}$ denotes the holomorphic tangent bundle. The spin bundles (E.3) exist if and only if the square root $K^{1/2}$ exists. A *natural* choice for $L$ on a Kähler manifold is thus $L = K^{-1}$. If we denote $\pi : \mathcal{M}_p \to S^2$ as the projection onto the bolt/zero-section, then for the natural complex structure on $\mathcal{M}_p$ implied by our notation we have

$$K = \pi^* \mathcal{O}(p-2) . \tag{E.4}$$

We thus see that $K^{1/2}$ indeed exists if and only if $p$ is even. The spinor bundles are (formally when $p$ is odd) hence

$$\begin{aligned} \mathcal{S}_+ &= \pi^* \left[ \mathcal{O}(\tfrac{p}{2} - 1) \oplus \mathcal{O}(-\tfrac{p}{2} + 1) \right] , \\ \mathcal{S}_- &= \pi^* \left[ \mathcal{O}(-\tfrac{p}{2} - 1) \oplus \mathcal{O}(\tfrac{p}{2} + 1) \right] . \end{aligned} \tag{E.5}$$

Since $L = \pi^* \mathcal{O}(n)$ by definition, we thus compute the spin$^c$ bundles

$$\begin{aligned} \mathcal{S}_+ \otimes L^{1/2} &= \pi^* \left[ \mathcal{O}(\tfrac{n+p}{2} - 1) \oplus \mathcal{O}(\tfrac{n-p}{2} + 1) \right] , \\ \mathcal{S}_- \otimes L^{1/2} &= \pi^* \left[ \mathcal{O}(\tfrac{n-p}{2} - 1) \oplus \mathcal{O}(\tfrac{n+p}{2} + 1) \right] . \end{aligned} \tag{E.6}$$

In particular, notice that since $n \equiv p \mod 2$, these bundles always exist on $\mathcal{M}_p$, as advertised. The Dirac spinors on our bolt solutions are globally sections of the bundles $\mathcal{S} \otimes L^{1/2} = \left(\mathcal{S}_+ \otimes L^{1/2}\right) \oplus \left(\mathcal{S}_- \otimes L^{1/2}\right)$, where the factors are given by (E.6) and $n$ is the flux number given by (E.2). Notice we have made use of (E.6) in the main text, for example to deduce (3.59).

Now we consider how these spinors restrict to the conformal boundary $S^3/\mathbb{Z}_p = \partial \mathcal{M}_p$. Denote the inclusion of this boundary as $\iota : S^3/\mathbb{Z}_p \hookrightarrow \mathcal{M}_p$. Then $H^2(S^3/\mathbb{Z}_p, \mathbb{Z}) \cong H_1(S^3/\mathbb{Z}_p, \mathbb{Z}) \cong \mathbb{Z}_p$, and the map

$$\mathbb{Z} \ \cong \ H^2(\mathcal{M}_p, \mathbb{Z}) \ \xrightarrow{\iota^*} \ H^2(S^3/\mathbb{Z}_p, \mathbb{Z}) \ \cong \ \mathbb{Z}_p \tag{E.7}$$

is simply reduction mod $p$. Let us denote the torsion line bundle that generates $H^2(S^3/\mathbb{Z}_p, \mathbb{Z}) \cong \mathbb{Z}_p$ by $\mathscr{L}$, so that $c_1(\mathscr{L}) = 1 \in \mathbb{Z}_p$. Then using (E.7) we can determine that the restriction of either spin$^c$ bundle to the conformal boundary is

$$\text{boundary spin}^c \text{ bundle } = \iota^*\mathcal{S}_\pm \otimes L^{1/2} = \mathscr{L}^{\frac{n+p}{2}} \otimes \left(\mathscr{L} \oplus \mathscr{L}^{-1}\right) . \qquad \text{(E.8)}$$

Here it is important to note that $\mathscr{L}^p = 1$ is a trivial line bundle, so that $\mathscr{L}^{\frac{n+p}{2}} = \mathscr{L}^{\frac{n-p}{2}}$. Thus the boundary spinors are typically sections of a non-trivial bundle.

Recall that every orientable three-manifold is spin, so a spin bundle of $S^3/\mathbb{Z}_p$ certainly exists. However, an important subtley here is that for $p$ odd there is a *unique* spin bundle, namely

$$\mathscr{S} = \mathscr{L} \oplus \mathscr{L}^{-1} , \qquad \text{(E.9)}$$

while for $p$ even there are *two* inequivalent spin bundles, namely

$$\mathscr{S}_0 = \mathscr{L} \oplus \mathscr{L}^{-1} , \qquad \mathscr{S}_1 = \mathscr{L}^{\frac{p}{2}+1} \oplus \mathscr{L}^{-\frac{p}{2}-1} . \qquad \text{(E.10)}$$

This arises from the fact that, quite generally, inequivalent spin bundles correspond to elements of $H^1(\mathcal{M}, \mathbb{Z}_2)$, and in the case at hand using the universal coefficient theorem one can compute $H^1(S^3/\mathbb{Z}_p, \mathbb{Z}_2) \cong \mathbb{Z}_{\mathrm{gcd}(p,2)}$. Thus for $p$ odd this group is trivial, while for $p$ even it is isomorphic to $\mathbb{Z}_2$. Concretely, when $p$ is even the two spinor bundles in (E.10) differ in that the spinors differ by a sign on going once around the Hopf fibre. We have then explicitly shown that the spin bundle $\mathscr{S}_1$ extends to either of the unique chiral spin bundles $\mathcal{S}_\pm$ over $\mathcal{M}_p$ in (E.5), while $\mathscr{S}_0$ extends instead to a particular spin$^c$ bundle on $\mathcal{M}_p$.[1]

The above discussion implies that a section of the spin bundle $\mathscr{S}_0$ is the same thing as a section of the spin$^c$ bundle $\mathscr{S}_1 \otimes \mathscr{L}^{\frac{p}{2}}$. This isomorphism is important for understanding the Killing spinors. Recall that in the 1/2 BPS case we always have $n = \pm p$. When $p$ is even the spinor bundles $\mathcal{S}_\pm$ restrict to $\mathscr{S}_1$ on the boundary, and it is precisely the flux $n = \pm p$ that turns this into the spinor bundle $\mathscr{S}_0$, as is clear from (E.8). At the level of the Killing spinor equation itself, the difference in the global form of the spin connection for $\mathscr{S}_0$ and $\mathscr{S}_1$ is equivalent to the difference between having no flat connection and the specific flat connection on $\mathscr{L}^{\frac{p}{2}}$. The reader might re-examine the (essentially local) discussion of the explicit spinors in section 3.2.3 in light of this global point. The 1/4 BPS case involves an additional subtlety, that we address in the next subsection E.2.

Finally, let us explain *why* (E.9), (E.10) are in fact spinor bundles for $S^3/\mathbb{Z}_p$! If we view $S^3/\mathbb{Z}_p$ as a $p$th power of the Hopf fibration over $S^2$, then this naturally leads to the tangent bundle being

$$T(S^3/\mathbb{Z}_p) = \mathbb{R} \oplus \mathscr{L}^2 , \qquad \text{(E.11)}$$

[1] The reader might be more familiar with this in the case of spinors on the circle $S^1$: there are two spin structures, periodic and anti-periodic. Only the anti-periodic choice extends to the spin structure on $\mathbb{R}^2$. It is similar here: it is the “anti-periodic” spinor bundle $\mathscr{S}_1$ that extends to a spinor bundle on $\mathcal{M}_p$.

where we have used that the tangent bundle for $S^2$ is $\mathcal{O}(2)$, and pulled this back to $S^3/\mathbb{Z}_p$ to obtain $\mathscr{L}^2$. The factor of $\mathbb{R}$ in (E.11) is tangent to the vector field $\partial_\psi$, generating the $S^1$ fibres. Given that the spinor bundle is a $\mathbb{C}^2$ vector bundle with structure group $SU(2)$, combined with the constraint that $\mathbb{P}(\mathscr{S}) = T_{\text{unit}}$, relating the projectivized spinor bundle to the bundle of unit tangent vectors, this implies that $\mathscr{S}$ must be of the form $P \oplus P^{-1}$ where $P$ is a line bundle satisfying $P^2 = \mathscr{L}^2$. This leads directly to (E.9) as the unique solution when $p$ is odd, and to the two solutions (E.10) when $p$ is even.

## E.2 Explicit computations

Guided by the above discussion, we may now look more closely at the local solutions to the Killing spinor equations in section 3.2.3.

### E.2.1 Flat connections

We first look more closely at the gauge field on $\mathcal{M}_p$, and in particular its global structure on the boundary. Suppose we have a gauge field on $\mathcal{M}_p$ given by

$$A \quad = \quad \kappa(r)(\mathrm{d}\psi + \cos\theta \mathrm{d}\varphi) \ , \tag{E.12}$$

where $\psi$ has period $4\pi/p$ and the bolt is at $r = r_0$. Flux quantization through this bolt gives

$$\int_{S^2_{r=r_0}} \frac{F}{2\pi} \quad = \quad -2\kappa(r_0) \ \equiv \ q \ . \tag{E.13}$$

Then $A$ is a connection on the line bundle $\mathcal{O}(q) \to \mathcal{M}_p$, where we are for now assuming that $q \in \mathbb{Z}$ is an integer, so that this makes sense. The expression (E.12) is ill-defined at $r = r_0$, where the vector field $\partial_\psi$ is zero. This is because $A$ cannot have an expression in terms of a global one-form on $\mathcal{M}_p$ when $q \neq 0$.

We remedy this as follows. Let $\theta$ and $\varphi$ be the standard coordinates on $S^2$, and cover this $S^2$ with coordinate patches $U_\pm$, in which $U_+$ excludes the south pole at $\theta = \pi$, and $U_-$ excludes the north pole at $\theta = 0$. On the products $U_\pm \times S^1_\pm$ we may define the one-forms

$$D\nu_\pm \quad \equiv \quad \mathrm{d}\nu_\pm + \frac{p}{2}(\cos\theta \mp 1)\mathrm{d}\varphi \ . \tag{E.14}$$

Here $\nu_\pm$ are coordinates on $S^1_\pm$, respectively, each with period $2\pi$. In order to form $S^3/\mathbb{Z}_p$, which are the constant $r > r_0$ surfaces, we then glue these together on the overlap via

$$\nu_+ - \nu_- \quad = \quad p\varphi \ . \tag{E.15}$$

Here the transition function $g : (0, \pi) \times S^1 \to U(1)$ is $g(\theta, \varphi) = \mathrm{e}^{\mathrm{i}p\varphi}$. This has winding number $p \in \mathbb{Z}$, and defines the principal $U(1)$ bundle over $S^2$ with first Chern class $p \in \mathbb{Z} \cong H^2(S^2, \mathbb{Z})$. Then on the overlap $D\nu_+ = D\nu_-$, and (E.14) defines the global angular form for the principal $U(1)$ bundle. Notice then that, in terms of the Euler angles used in the main text,

$$\nu_\pm \quad = \quad \frac{p}{2}\psi_\pm \ , \tag{E.16}$$

and the globally defined one-form defined by (E.14) is simply $\frac{p}{2}\sigma_3$.

We may then cover our manifold $\mathcal{M}_p$ by the two coordinate patches $\mathbb{R}_{\geq 0} \times U_\pm \times S^1_\pm$, where $r - r_0$ is a coordinate on $\mathbb{R}_{\geq 0}$. Then in these two patches we define

$$A_\pm \quad = \quad \frac{q}{2}\mathrm{d}\psi_\pm + \kappa(r)(\mathrm{d}\psi_\pm + (\cos\theta \mp 1)\mathrm{d}\varphi) \ . \tag{E.17}$$

This is the correct non-singular form of (E.12) in each coordinate patch. Moreover, on the overlap in $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p = \{r > r_0\}$ (notice it is crucial here that we exclude the bolt at $r = r_0$) we have

$$A_+ - A_- \quad = \quad q\mathrm{d}\varphi \ . \tag{E.18}$$

It follows that on the complement of the bolt $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_p$ we may write

$$A \quad = \quad \kappa(r)\sigma_3 + A^{(3)}_{\text{flat}} \ , \tag{E.19}$$

where $A^{(3)}_{\text{flat}}$ is a *flat connection* on $\mathscr{L}^q$, where $\mathscr{L}$ has first Chern class $c_1(\mathscr{L}) = 1 \in \mathbb{Z}_p \cong H^2(S^3/\mathbb{Z}_p, \mathbb{Z})$. This is defined in the two patches

$$A^{(3)}_{\text{flat}} \quad = \quad \begin{cases} \frac{q}{2}\mathrm{d}\psi_+ & \text{in } U_+ \times S^1_+ \\ \frac{q}{2}\mathrm{d}\psi_- & \text{in } U_- \times S^1_- \end{cases} \ . \tag{E.20}$$

This is manifestly flat, and on the overlap we have

$$A^{(3)}_{\text{flat},+} - A^{(3)}_{\text{flat},-} \quad = \quad q\mathrm{d}\varphi \ , \tag{E.21}$$

which is indeed precisely the transition function that defines $\mathscr{L}^q$. The holonomy of this connection around any $S^1$ fibre in $S^3/\mathbb{Z}_p$ is

$$\exp\left(\mathrm{i}\int_{S^1_{\text{fibre}}} A^{(3)}_{\text{flat}}\right) \quad = \quad \exp\left(\frac{2\pi \mathrm{i} q}{p}\right) \ , \tag{E.22}$$

which is the observable Wilson line of this non-trivial connection.

What we have shown here, very explicitly, is that if the gauge field is a connection on $\mathcal{O}(q) \to \mathcal{M}_p$, which has first Chern class $c_1(\mathcal{O}(q)) = q \in \mathbb{Z} \cong H^2(\mathcal{M}_p, \mathbb{Z})$, then the restriction of this first Chern class to the boundary $S^3/\mathbb{Z}_p$ is simply $q \bmod p$ in

$H^2(S^3/\mathbb{Z}_p,\mathbb{Z}) \cong \mathbb{Z}_p$. Topologically this is clear, since the natural map

$$\mathbb{Z} \;\cong\; H^2(\mathcal{M}_p,\mathbb{Z}) \;\to\; H^2(S^3/\mathbb{Z}_p,\mathbb{Z}) \;\cong\; \mathbb{Z}_p\ , \tag{E.23}$$

is just reduction mod $p$.

When $q$ is half-integer, which happens when $p$ is odd and $A$ is a spin$^c$ connection, the above discussion cannot be applied directly. For example, for the 1/2 BPS solutions we have $q = \pm\frac{p}{2}$. In particular, the transition function (E.18) is not a single-valued $U(1)$ gauge transformation in this case. One might proceed in this case by multiplying the gauge field by 2, and note that $2q = \pm p = 0$ mod $p$, and then that when $p$ is odd the only solution to $2q = 0$ mod $p$ is $q = 0$. Thus the boundary torsion is zero in this case. Although slightly indirect, this is a perfectly valid argument to reach this conclusion, which we have then used in the main text. A more direct proof, using coordinate patches, requires a more involved explicit treatment than we have given above.

### E.2.2 Boundary spinors

With this in hand, we can return to the explicit boundary Killing spinors in section 3.2.3. Beginning with the 1/2 BPS case, the explicit solution to the Killing spinor equation is (3.29). We first note that the frame $\tilde{e}^a$ in (3.28) is not invariant under $\mathcal{L}_{\partial_\psi}$, but rather

$$\begin{pmatrix} \tilde{e}^1 \\ \tilde{e}^2 \\ \tilde{e}^3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\theta \\ \sin\theta\mathrm{d}\varphi \\ 2s\sigma_3 \end{pmatrix} . \tag{E.24}$$

Here $\sigma_3$ is globally defined on $S^3/\mathbb{Z}_p$, being $\frac{2}{p}Dv_\pm$ in each patch given by (E.14). The $SO(3)$ rotation above corresponds to the $SU(2) = \mathrm{Spin}(3)$ rotation

$$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix} , \tag{E.25}$$

so that in the frame $\check{e}^1 = \mathrm{d}\theta$, $\check{e}^2 = \sin\theta\mathrm{d}\varphi$, $\check{e}^3 = 2s\sigma_3$ the spinor (3.29) reads

$$\check{\chi} = \begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi/2} & -\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi/2} \\ \gamma\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi/2} & \gamma\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi/2} \end{pmatrix} \chi_{(0)} . \tag{E.26}$$

This is independent of $\psi$, as claimed. However, the frame $\check{e}^a$ is *singular* at the poles $\theta = 0$, $\theta = \pi$, which are coordinate singularities. In the patch $U_+ \times S^1_+$, which recall excludes the south pole $\theta = \pi$, we may further rotate the frame to

$$\begin{pmatrix} e^1_+ \\ e^2_+ \\ e^3_+ \end{pmatrix} \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \check{e}^1 \\ \check{e}^2 \\ \check{e}^3 \end{pmatrix} \sim \begin{pmatrix} \mathrm{d}x_+ \\ \mathrm{d}y_+ \\ 2s\sigma_3 \end{pmatrix} , \tag{E.27}$$

where we have defined $x_+ = \theta\cos\varphi$, $y_+ = \theta\sin\varphi$, and the last equality is true to leading order near to $\theta = 0$. Near to $\theta = 0$, these are standard Cartesian coordinates on $\mathbb{R}^2$, with $\theta$ playing the role of the usual radial coordinate. Thus the frame $e^a_+$ is non-singular in the patch $U_+ \times S^1_+$, and the corresponding spinor rotates similarly to (E.25) to give[2]

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \\ \gamma\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi} & \gamma\cos\frac{\theta}{2} \end{pmatrix} \chi_{(0)} . \tag{E.28}$$

We see that this is indeed smooth in this patch, the point being that the terms $\mathrm{e}^{-\mathrm{i}\varphi}$, which are ill-defined at $\theta = 0$, have coefficients which vanish as $\mathcal{O}(\theta)$ at $\theta = 0$.

A similar argument now works in the south patch $U_- \times S^1_-$, with $x_- = -(\pi - \theta)\cos\varphi$, $y_- = (\pi - \theta)\sin\varphi$. The rotation then has the *opposite* sign to (E.27),

$$\begin{pmatrix} e^1_- \\ e^2_- \\ e^3_- \end{pmatrix} \equiv \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \check{e}^1 \\ \check{e}^2 \\ \check{e}^3 \end{pmatrix} \sim \begin{pmatrix} \mathrm{d}x_- \\ \mathrm{d}y_- \\ 2s\sigma_3 \end{pmatrix} , \tag{E.29}$$

leading to the corresponding spinor in the corresponding smooth frame $e^a_-$

$$\chi_- = \begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi} & -\sin\frac{\theta}{2} \\ \gamma\sin\frac{\theta}{2} & \gamma\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \chi_{(0)} . \tag{E.30}$$

This is then smooth in the patch $U_- \times S^1_-$.

Our spinor is thus smooth in each coordinate patch of $S^3/\mathbb{Z}_p$, and on the overlap region they are related by the $U(1) \subset SU(2) \cong \mathrm{Spin}(3)$ transformation

$$\chi_- = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \chi_+ . \tag{E.31}$$

This precisely means that, globally, the spinors are sections of $\mathscr{L} \oplus \mathscr{L}^{-1}$, precisely as we claimed using more abstract reasoning in section E.1. We have thus checked that the 1/2 BPS spinors are globally well-defined and smooth on the constant $r > r_0$ surfaces $S^3/\mathbb{Z}_p$, and sections of the bundle $\mathscr{S}$ in (E.9) and $\mathscr{S}_0$ in (E.10), when $p$ is odd and even, respectively.

The story for the 1/4 BPS spinors is very similar, with just one important difference.

[2] Here $\chi_+$ denotes the spinor $\chi$ in the patch $U_+ \times S^1_+$, and is not to be confused with the use of $\pm$ in section 3.2.3 to denote chirality!

Although the spinor (3.37) is simply constant in the frame $\tilde{e}^a$, because the latter depends on $\psi$ as in (E.24) in fact the 1/4 BPS spinors are charged under $\partial_\psi$. Specifically, (3.37) satisfies

$$\mathcal{L}_{\partial_\psi}\chi = \frac{\mathrm{i}}{2}\chi \ , \tag{E.32}$$

implying an overall phase dependence of $\mathrm{e}^{\mathrm{i}\psi/2}$. This would then seem problematic if one tries to take $\psi$ to have period $4\pi/p$ for general $p > 1$. However, we emphasized in section 3.2.3 that the computation was only valid *locally*, and indeed for the 1/4 BPS Quaternionic-Eguchi-Hanson solutions in section 3.3 (and of course the more general 1/4 BPS solutions in section 3.5), the gauge field flux (3.57) implies that on $S^3/\mathbb{Z}_p$ we have an additional flat connection on $\mathscr{L}^{-1}$. This flat connection is given explicitly in coordinate patches by (E.20), (E.21), with $q = -1$. If one *includes* this gauge field when solving for the 1/4 BPS Killing spinors in each patch, then one obtains an *additional* phase dependence of $\mathrm{e}^{-\mathrm{i}\psi_\pm/2}$. This phase then *cancels* the phase arising from (E.32), and the upshot is that the global 1/4 BPS spinor is in fact *independent of* $\psi$. We thus see that the $-1$ factor in the quantized flux (3.57) and (3.113) is crucial for supersymmetry for general $p > 1$.

Including this flat connection, then in the frame $\breve{e}^1 = \mathrm{d}\theta$, $\breve{e}^2 = \sin\theta\mathrm{d}\varphi$, $\breve{e}^3_\pm = 2s(\mathrm{d}\psi_\pm + (\cos\theta \mp 1)\mathrm{d}\varphi)$, one find that the 1/4 BPS spinors in the two patches are explicitly

$$\breve{\chi}_\pm = \mathrm{e}^{\mp\mathrm{i}\varphi/2}\begin{pmatrix} 0 \\ \chi^{(-)}_{(0)} \end{pmatrix} \ . \tag{E.33}$$

Rotating as in (E.27) and (E.29) in each patch, to give smooth frames $e^a_\pm$ as before, one then sees that these 1/4 BPS spinors on constant $r > r_0$ surfaces $S^3/\mathbb{Z}_p$ are smooth sections of $(\mathscr{L} \oplus \mathscr{L}^{-1}) \otimes \mathscr{L}^{-1}$.

### E.2.3 Regularity at the bolt

The above discussion guarantees that the spinors are well-defined and smooth on $\{r > r_0\}$, where the bolt $S^2$ is at $r = r_0$. For completeness, we should also verify that the spin$^c$ spinors in section 3.2.3 are smooth at the bolt itself.

This is easily checked along the lines of the previous subsection. We first note that the four-frame (3.14) is *singular* at the bolt $r = r_0$ itself, and moreover the gauge field in (3.6) is also singular at the bolt. Thus the spinors in section 3.2.3 are in a singular frame, in a singular gauge! However, this is easily rectified by making an appropriate frame rotation and gauge transformation, respectively.

If we denote by $\rho$ the geodesic distance from the bolt at $r = r_0$, then to leading order near $\rho = 0$ the frame (3.14) reads

$$
\begin{aligned}
e^1 &\sim \sqrt{r_0^2 - s^2}\sigma_1 \ , \quad e^2 \sim \sqrt{r_0^2 - s^2}\sigma_2 \ , \quad e^3 \sim \rho\left[\mathrm{d}\left(\frac{p\psi}{2}\right) + \frac{p}{2}\cos\theta\mathrm{d}\varphi\right] \ , \\
e^4 &\sim \mathrm{d}\rho \ ,
\end{aligned}
\tag{E.34}
$$

as in equation (3.70). The $e^3$ and $e^4$ directions suffer the same polar coordinate type singularity at $\rho = 0$ as the frame $\check{e}^a$ suffered at $\theta = 0$, $\theta = \pi$ in the previous subsection. If we rotate

$$
\begin{pmatrix} e_0^1 \\ e_0^2 \\ e_0^3 \\ e_0^4 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{p\psi}{2} & \sin\frac{p\psi}{2} \\ 0 & 0 & -\sin\frac{p\psi}{2} & \cos\frac{p\psi}{2} \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} \ , \tag{E.35}
$$

the $e_0^3$, $e_0^4$ are now smooth near the bolt. The corresponding action on the Dirac spinors may be deduced from the four-dimensional gamma matrices (3.16), and is

$$
\mathrm{diag}(\mathrm{e}^{-\mathrm{i}p\psi/4}, \mathrm{e}^{\mathrm{i}p\psi/4}, \mathrm{e}^{\mathrm{i}p\psi/4}, \mathrm{e}^{-\mathrm{i}p\psi/4}) \in \mathrm{Spin}(4) \cong SU(2) \times SU(2) \ . \tag{E.36}
$$

Of course, we should again introduce coordinate patches $U_\pm$ on the $S^2$ bolt, and rotate the $e_0^1$ and $e_0^2$ directions precisely as we did in the previous section, *i.e.* we apply the rotation (E.24) so that the frame is invariant under $\mathcal{L}_{\partial_\psi}$, and the rotations (E.27), (E.29) in the $U_+$ and $U_-$ patches, respectively. In this way we obtain four-frames $e_\pm^a$, $a = 1, 2, 3, 4$, in patches $U_\pm \times S_\pm^1 \times \mathbb{R}_{\geq 0}$ which cover a neighbourhood of the bolt. Here $\rho \in \mathbb{R}_{\geq 0}$ is geodesic distance from the bolt. In this frame, the 1/2 BPS spinors (3.23) read

$$
\epsilon = \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\chi^{(+)}\mathrm{e}^{-\mathrm{i}p\psi/4} \\ \sqrt{\frac{(r-r_1)(r-r_2)}{r-s}}\chi^{(-)}\mathrm{e}^{\mathrm{i}p\psi/4} \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\chi^{(+)}\mathrm{e}^{\mathrm{i}p\psi/4} \\ \mathrm{i}\sqrt{\frac{(r-r_3)(r-r_4)}{r+s}}\chi^{(-)}\mathrm{e}^{-\mathrm{i}p\psi/4} \end{pmatrix} \ , \tag{E.37}
$$

where $\chi^{(\pm)}$ are the two components of $\chi$ in (E.28) and (E.30), in the two patches respectively. Similarly, one should understand $\psi = \psi_\pm$ in the two patches, respectively.

Finally, recall that the gauge for the spin$^c$ gauge field $A$ is singular at the bolt, as discussed in section E.2.1. For the positive/negative branch 1/2 BPS solutions, the singular gauge field is to leading order

$$
A \sim \mp\frac{p}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi) \ , \tag{E.38}
$$

near the bolt, respectively. This follows directly from (3.81). Thus for the positive/negative branch solutions we must make a gauge transformation $A \to A \pm \frac{p}{4}\mathrm{d}\psi$ (in each patch appropriately) in order that $A$ is well-defined at the bolt (where the azimuthal coordinate $\psi$ is not defined). Doing so, we obtain the following form of the spinors for the positive branch solutions

$$\epsilon_{\text{positive branch}} = \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\chi^{(+)} \\ \sqrt{\frac{(r-r_1)(r-r_2)}{r-s}}\chi^{(-)}\mathrm{e}^{\mathrm{i}p\psi/2} \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\chi^{(+)}\mathrm{e}^{\mathrm{i}p\psi/2} \\ \mathrm{i}\sqrt{\frac{(r-r_3)(r-r_4)}{r+s}}\chi^{(-)} \end{pmatrix}, \tag{E.39}$$

while the negative branch spinors are

$$\epsilon_{\text{negative branch}} = \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{r-s}}\chi^{(+)}\mathrm{e}^{-\mathrm{i}p\psi/2} \\ \sqrt{\frac{(r-r_1)(r-r_2)}{r-s}}\chi^{(-)} \\ \mathrm{i}\sqrt{\frac{(r-r_1)(r-r_2)}{r+s}}\chi^{(+)} \\ \mathrm{i}\sqrt{\frac{(r-r_3)(r-r_4)}{r+s}}\chi^{(-)}\mathrm{e}^{-\mathrm{i}p\psi/2} \end{pmatrix}. \tag{E.40}$$

These spinors are now in a non-singular frame and gauge at the bolt, and we indeed see that they are smooth. Here one must recall that for the positive branch the bolt is at $r_0 = r_2$, while for the negative branch instead $r_0 = r_4$. In both cases $r_0$ is the largest root, so $r > s$ for all $r$ while $r > r_i$ provided $r_i$ is not the root $r_0$. The key point is that for the positive branch spinor (E.39), the components that depend on $\psi$ tend to zero at the bolt $r = r_2$, with a corresponding statement holding for (E.40). Indeed, notice that $p\psi/2$ has the canonical period $2\pi$, with geodesic distance $\rho \propto \sqrt{r - r_0}$ near the bolt, so that the spinors tend to zero near the bolt in the same way as they tend to zero near the poles $\theta = 0$, $\theta = \pi$ in (E.28), (E.30), respectively. This proves that the 1/2 BPS spin$^c$ spinors are smooth and well-defined everywhere, for both positive and negative branches.

The discussion for the 1/4 BPS case is essentially identical (although here notice that our labelling of roots $r_4 \leftrightarrow r_2$ for the two types of branch is interchanged relative to the 1/2 BPS case).

## E.3 Eleven-dimensional spinors

In this appendix we briefly consider the eleven-dimensional spinors for the bolt solutions. Even though the four-dimensional Taub-Bolt-AdS solutions are not spin manifolds for $p$ odd, we will see that the eleven-dimensional Euclidean space is always spin, and that the eleven-dimensional spinors are indeed globally well-defined whenever the metric is. We follow the notation of section 3.6.

We consider the case of lifting a Taub-Bolt-AdS solution, with topology $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$, on a regular Sasaki-Einstein manifold $Y_7$ with Kähler-Einstein base $B_6$, Fano index $I = I(B_6)$, and for simplicity we take $k = I$ so that $Y_7$ is the total space of the $U(1)$ principal bundle associated to the canonical bundle of $B_6$. In this case from (3.121) we see that the eleven-dimensional geometry is the total space of a $U(1)$ principal bundle over $\mathcal{M}_p \times B_6$, with global angular form $\eta + \frac{1}{2}A$. We denote the corresponding line bundle by $\mathcal{V}$. We will show that the total space $Z$ of $\mathcal{V}$ (which is twelve-dimensional) is always a spin manifold. Since $Z$ deformation retracts onto its zero section, it is sufficient to compute the restriction of $w_2(Z)$ to the zero section $\mathcal{M}_p \times B_6$. In turn, we note that $Z$ has a natural complex structure (with $\mathcal{M}_p$ having the complex structure of section E.1), and then $w_2(Z)$ is the mod 2 reduction of the first Chern class $c_1(Z)$. We then compute $c_1(\mathcal{M}_p \times B_6) = (2-p)\Phi + c_1(B_6)$, where $\Phi$ denotes the generator of $H^2(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$.[3] Then the connection term $\eta + \frac{1}{2}A$ implies that $c_1(\mathcal{V}) = -n\Phi - c_1(B_6)$. The Whitney product formula then gives $c_1(Z) = (2-p-n)\Phi$ (with $\Phi$ understood as appropriately pulled back). Since $p \equiv n$ mod 2, we see that $c_1(Z) = 0$ mod 2, which implies that $Z$ is indeed a spin manifold. Its eleven-dimensional boundary, which is our spacetime, is thus also spin.

The connection term $\eta + \frac{1}{2}A$ is thus precisely ensuring that the eleven-dimensional spacetime is a spin manifold, even though the base four-dimensional spacetime in general is not. This term also plays an important role in ensuring that the eleven-dimensional spinor is indeed a spinor, rather than a section of a spin$^c$ bundle. The eleven-dimensional spinor is a tensor product $\epsilon \otimes \beta$, where $\epsilon$ is the Dirac spin$^c$ spinor on $\mathcal{M}_p$, and $\beta$ is a spinor on the internal space $Y_7$. In particular, $\epsilon$ is coupled to the spin$^c$ line bundle $L^{1/2}$, with (formal) connection $A$. However, because of the connection term $\eta + \frac{1}{2}A$ the spinor $\beta$ is also fibred over $\mathcal{M}_p$. To see this, note that on a Sasaki-Einstein seven-manifold the Killing spinor has charge 2 under $\partial_\xi$, where recall $\eta = \mathrm{d}\xi + \sigma$. Thus the additional connection term in $\eta + \frac{1}{2}A$ implies that $\beta$ has charge $-1$ under $A$. Thus $\beta$ is a spinor on $Y_7$, but also valued in $L^{-1/2}$. Altogether, we see that the dependence on $L$ cancels in the tensor product $\epsilon \otimes \beta$, which precisely ensures that this is then an eleven-dimensional spinor, rather than a spin$^c$ spinor. As we have seen in the previous paragraph, this is then guaranteed to be globally defined.

[3]That is, $\int_{S^2_{\text{bolt}}} \Phi = 1$.

# F | Free energy

## F.1 Holographic renormalization

In this appendix we present some further details of the computation of the holographic free energy/Euclidean action of the solutions described in the main text.

We begin by writing the supergravity action

$$I \;\equiv\; I^{\text{grav}}_{\text{bulk}} + I^{F} \;=\; -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\,(R+6) + \frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\,F^2 \;. \tag{F.1}$$

This action diverges as $r \to \infty$ and in order to obtain a finite value we apply the standard technique of holographic renormalization [50, 51]. We introduce a cut-off at $r = \varrho$ and consider the hypersurface $\mathcal{S}_\varrho$ of constant $r = \varrho$ with induced metric

$$\gamma_{\mu\nu} = g_{\mu\nu} - \hat{n}_\mu \hat{n}_\nu \;, \tag{F.2}$$

where $\hat{n}$ is the unit vector normal to $\mathcal{S}_\varrho$. As $\varrho \to \infty$, $\mathcal{S}_\varrho$ becomes the (conformal) boundary and $\gamma_{\mu\nu}$ the boundary metric. We regularize the action by adding the following term

$$I^{\text{grav}}_{\text{ct}} + I^{\text{grav}}_{\text{bdry}} \;=\; \frac{1}{8\pi G_4}\int_{\mathcal{S}_\varrho} \mathrm{d}^3x\sqrt{\gamma}\left(2 + \tfrac{1}{2}R(\gamma) - K\right) \;. \tag{F.3}$$

Here $R(\gamma)$ is the Ricci scalar of $\gamma_{\mu\nu}$, and $K$ is the trace of the second fundamental form of $\mathcal{S}_\varrho$, the latter being the Gibbons-Hawking boundary term.

## F.2 Proof that $I_{\text{sing}} = \frac{n^2\pi}{8pG_4}$

The space Taub-NUT-AdS$/\mathbb{Z}_p$ is a singular orbifold for $p > 1$. We have seen in our explicit examples that Taub-NUT-AdS$/\mathbb{Z}_p$ solutions can arise, with specific squashing parameters, as limits of Taub-Bolt-AdS solutions. When this happens, the *singularity* of the Taub-NUT-AdS$/\mathbb{Z}_p$ can effectively contribute to the free energy. This is because, in these limits, the Taub-NUT-AdS$/\mathbb{Z}_p$ solution has an additional flat gauge field turned on, which can be understood as originating from "trapped flux" at the bolt which has collapsed to zero size. In this appendix we attempt to understand this phenomenon more generally. We argue that the singularity can contribute to the free energy via

$$I_{\text{sing}} \equiv \frac{n^2}{8p}\cdot\frac{\pi}{G_4} \ . \tag{F.4}$$

The basic physical idea here is that the singularity can have $\frac{n}{2}$ units of flux “trapped” in it, so that

$$\int_{\text{collapsed cycle}} \frac{F}{2\pi} = \frac{n}{2} \ . \tag{F.5}$$

This flux then induces a corresponding torsion line bundle on Taub-NUT-AdS/$\mathbb{Z}_p$ minus the singularity, which has topology $\mathbb{R}_{>0}\times S^3/\mathbb{Z}_p$. In practice, we compute this singular contribution by choosing a one-parameter family of resolutions of the orbifold singularity to $\mathcal{M}_p = \mathcal{O}(-p)\to S^2$, and then calculating the free energy of an appropriate gauge field satisfying (F.5), where the collapsed cycle is resolved to the $S^2$ bolt/zero-section. This one-parameter family, depending on $\varepsilon > 0$, will be such that in the $\varepsilon\to 0$ limit we recover the Taub-NUT-AdS/$\mathbb{Z}_p$ metric with a *flat* torsion gauge field, with line bundle depending on $n$. The result will end up being a topological invariant, provided we make certain natural assumptions.[1]

We begin by first choosing an explicit resolution of the metric and appropriate gauge field, which will lead to (F.4). Having done this, we will then discuss to what extent the result is independent of these choices, and why (F.4) may then be interpreted as a topological invariant.

Recall that the self-dual Einstein metric on the Taub-NUT-AdS space can be written as

$$\mathrm{d}s_4^2 = \frac{r^2-s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2-s^2)(\sigma_1^2+\sigma_2^2) + \frac{4s^2\Omega(r)}{r^2-s^2}\sigma_3^2 \ , \tag{F.6}$$

where

$$\sigma_1 + \mathrm{i}\sigma_2 = \mathrm{e}^{-\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi) \ , \quad \sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi \ . \tag{F.7}$$

Here $\Omega(r) = (r-s)^2(r-r_1)(r-r_2)$, where

$$\left\{\begin{array}{c} r_2 \\ r_1 \end{array}\right\} = \left\{\begin{array}{c} -s+\sqrt{4s^2-1} \\ -s-\sqrt{4s^2-1} \end{array}\right\} \ . \tag{F.8}$$

Taking $\theta\in[0,\pi]$ and the periodicities $\varphi\in[0,2\pi)$, $\psi\in[0,4\pi)$ this space is topologically $\mathbb{R}^4$. Taking instead $\psi\in[0,\frac{4\pi}{p})$ this becomes topologically the orbifold $\mathbb{R}^4/\mathbb{Z}_p$, with a (NUT) orbifold singularity located at $r=s$.

To compute $I_{\text{sing}}$ we resolve the singularity, replacing it with an $S^2_\varepsilon$ of radius propor-

[1] Notice that the naive contribution of a flat torsion gauge field to the free energy is zero (because $F=0$).

tional to a small parameter $\varepsilon$, for any value of $s$. Obviously, we cannot do this while preserving supersymmetry and $SU(2)\times U(1)$ isometry in general, otherwise we would have found this metric within some class of BPS solutions. However, it is straightforward to write a metric on the resolved space that has the same isometry group and with same conformal boundary. A simple example of such a metric is obtained by replacing[2] $\Omega(r)$ with

$$\Omega^{\varepsilon}(r) \;\; = \;\; (r-s-\varepsilon)(r-s-a\varepsilon)(r-r_1-\varepsilon)(r-r_2-\varepsilon)\ , \tag{F.9}$$

where we assume that $\varepsilon \geq 0$. Notice that the roots are now all distinct, with the largest root being $r^{\varepsilon}=s+\varepsilon$, provided that $a<1$. Using the method described in the text, it is straightforward to check that taking

$$a \;\; = \;\; 1-p+\mathcal{O}(\varepsilon)\ , \tag{F.10}$$

this gives a *smooth* metric on the space $\mathcal{M}_p^{\varepsilon}=\mathcal{O}(-p)\to S^2$, for any value of $s$ and sufficiently small $\varepsilon>0$. Notice that indeed $a<1$ for any $p$, thus $r^{\varepsilon}$ is the largest root. Then $\Omega^{\varepsilon}(r)$ reduces smoothly to the Taub-NUT-AdS metric function for $\varepsilon\to 0$, where two roots coalesce.

In order to compute the contribution to the free energy of the trapped flux, we will choose a one-parameter family of gauge fields on this resolved space $\mathcal{M}_p^{\varepsilon}$ with self-dual field strength $F^{\varepsilon}$. Recall that locally the most general (anti-)self-dual gauge field preserving the isometry of the metric (F.6) is given by $A^{\pm}=C_{\pm}f_{\pm}(r)\sigma_3$, where $C_{\pm}$ are constants and

$$f_{\pm}(r) \;\; = \;\; \frac{r\mp s}{r\pm s}\ . \tag{F.11}$$

It turns out that choosing the (local) gauge field

$$A^{\varepsilon} \;\; = \;\; -\frac{n\varepsilon}{4(2s+\varepsilon)}\,\frac{r+s}{r-s}\,\sigma_3\ , \tag{F.12}$$

the flux through the $S^2_{\varepsilon}\subset\mathcal{M}_p^{\varepsilon}$ at $r=s+\varepsilon$ is the desired one, namely

$$\int_{S^2_{\varepsilon}}\frac{F^{\varepsilon}}{2\pi} \;\; = \;\; \frac{n}{2}\ , \tag{F.13}$$

again independently of $s$ and $\varepsilon$. Moreover $F^{\varepsilon}\to 0$ for $\varepsilon\to 0$ implying that, *globally*, $A^{\varepsilon}$ becomes a flat torsion gauge field in the limit. Finally, it is straightforward to compute

[2]We have checked that for any choice of parameters in $\Omega^{\varepsilon}(r)=(r-s-\varepsilon)(r-s-a\varepsilon)(r-r_1-b\varepsilon)(r-r_2-c\varepsilon)$ the resulting metric is not Einstein. However, this is not an issue, as will become apparent.

the contribution to the action/free energy

$$\begin{aligned}
\frac{1}{16\pi G_4}\int_{\mathcal{M}_p}(F^\varepsilon)^2 &= -\frac{1}{8\pi G_4}\int_{\mathcal{M}_p}F^\varepsilon\wedge F^\varepsilon \\
&= \frac{1}{8\pi G_4}\frac{n^2\varepsilon^2}{16(2s+\varepsilon)^2}[f_-(r=r^\varepsilon)^2-f_-(r=\infty)^2]\int \mathrm{d}\psi\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{n^2}{8p}\cdot\frac{\pi}{G_4}+\mathcal{O}(\varepsilon^2)\ , \qquad \text{(F.14)}
\end{aligned}$$

where in the last equality we used the fact that $\psi\in[0,\frac{4\pi}{p})$. We have thus derived (F.4), as advertised.

Although this result depends *a priori* on the choice of resolved metric and gauge field we picked, we will now explain to what extent it is in fact *independent* of these choices. Having resolved the singularity to $\mathcal{M}_p=\mathcal{O}(-p)\to S^2$, more generally we may consider any one-parameter family of gauge fields on this space, depending on $\varepsilon>0$, which satisfy the following properties: (i) the curvature $F^\varepsilon$ has finite action, (ii) $\frac{F^\varepsilon}{2\pi}$ has period $\frac{n}{2}$ through the $S^2$ bolt/zero-section, (iii) the curvature tends to zero in the Taub-NUT-AdS/$\mathbb{Z}_p$ space as $\varepsilon\to 0$ (say, $\mathcal{O}(\varepsilon)$). These are all clearly necessary (or at least reasonable) assumptions. In order to compute the contribution of this gauge field to the free energy, we will also assume that (iv) $F^\varepsilon$ satisfies the gauge field equation of motion. Of course, all these conditions are satisfied in our computation above.

With these assumptions in place, the integral $\int_{\mathcal{M}_p}F^\varepsilon\wedge *F^\varepsilon$ is in fact independent of the cohomology class of $F^\epsilon$, to leading order (*i.e.* ignoring $\mathcal{O}(\varepsilon)$ corrections). This follows by taking $F^\varepsilon\to F^\varepsilon+\mathrm{d}\Lambda$, where $\Lambda$ is any closed form, using the equation of motion for $F^\epsilon$, Stokes' Theorem, and the fact that the curvature is $\mathcal{O}(\varepsilon)$ at infinity. We may thus, without loss of generality, pick the particular representation (F.12) for this cohomology class. So far we have not specified what metric we are using to define the Hodge dual, but notice that since the one-parameter family of metrics is required to tend to the Taub-NUT-AdS metric as $\varepsilon\to 0$, and since our choice of gauge field (F.12) becomes (anti-)self-dual in this limit, without loss of generality we may pick an (anti-)self-dual gauge field for all $\varepsilon$. Essentially, any other choice will simply change only the $\mathcal{O}(\varepsilon)$ corrections to the final action/free energy integral.

The advantage of choosing an anti-self-dual field strength is that this makes it clear why the final result (F.4) is a topological invariant (even though the above argument shows that picking an anti-self-dual field strength is not necessary). We have, as before,

$$\frac{1}{16\pi G_4}\int_{\mathcal{M}_p}(F^\varepsilon)^2 = -\frac{1}{8\pi G_4}\int_{\mathcal{M}_p}F^\varepsilon\wedge F^\varepsilon\ . \qquad \text{(F.15)}$$

The right hand side may then be understood topologically, to leading order in $\varepsilon$, as the pairing $H^2_{\mathrm{cpt}}(\mathcal{M}_p,\mathbb{R})\times H^2(\mathcal{M}_p,\mathbb{R})\to\mathbb{R}$. Although $F^\varepsilon$ is *not* necessarily compactly supported (and is not in our example computation), it *is* to leading order in $\varepsilon$. We have

$H^2_{\rm cpt}(\mathcal{M}_p, \mathbb{Z}) \cong \mathbb{Z}$, and the generator $\Psi$ has unit integral over a fibre of $\mathcal{M}_p = \mathcal{O}(-p) \to S^2$ (it is the Thom class of this bundle). It is then a standard fact that $\int_{S^2} \Psi = -p$, the latter being the Euler class of the bundle $\mathcal{O}(-p) \to S^2$, so that $\int_{\mathcal{M}_p} \Psi \wedge \Psi = -p$ (integrating first over the fibre, and then over the bolt). Thus the cohomology class $[F^\varepsilon] = -\frac{\pi n}{p}\Psi$, and we hence compute

$$\begin{aligned} \frac{1}{16\pi G_4}\int_{\mathcal{M}_p} (F^\varepsilon)^2 &= -\frac{1}{8\pi G_4}\left(\frac{\pi n}{p}\right)^2 \int_{\mathcal{M}_p} \Psi \wedge \Psi \ , \\ &= \frac{n^2}{8p} \cdot \frac{\pi}{G_4} \ . \end{aligned} \tag{F.16}$$

Here each equality should be understood as up to $\mathcal{O}(\varepsilon)$. This explains why (F.4) may be understood as a topological invariant.

# G | Holographic Wilson loops

In this appendix we present an argument showing that the Taub-NUT-AdS and Taub-Bolt-AdS solutions behave *qualitatively* differently with respect to the holographic computation of the VEV of a BPS Wilson loop. Given a specific dual field theory Lagrangian, the latter is in principle computable (at finite $N$) using localization methods.

We consider an M2-brane that wraps the M-theory circle together with a copy of $\mathbb{R}^2 \subset \mathcal{M}^{(4)}$ that has boundary an $S^1 \subset S^3$ at conformal infinity. This naturally corresponds to a Wilson loop in the boundary gauge theory. Notice that, from the IIA point of view, this is a fundamental string wrapping the copy of $\mathbb{R}^2$. Taking the $S^1 \subset S^3$ to be a Hopf fibre/great circle, which in our coordinate system is coordinatized by the Euler angle $\psi$, and the $\mathbb{R}^2$ to be this together with the radial direction coordinatized by $r$ at $\theta = 0$, we conjecture that the wrapped string should be BPS, as it is in $\mathrm{AdS}_4$.[1] For a Taub-Bolt-AdS solution, notice this is a copy of the fibre of $\mathcal{M}^{(4)} = \mathcal{O}(-1) \to S^2$.

The action of the M2-brane/fundamental string should compute the VEV of the corresponding BPS Wilson loop in the holographically dual supersymmetric gauge theory, to leading order in the large $N$ limit. It is easy enough to compute this action in any particular example. The VEV of a BPS Wilson loop can also be computed *exactly* via localization in the gauge theory.

However, there is an important subtlety in this computation, for which the Taub-NUT-AdS space and Taub-Bolt-AdS space behave very differently. This was first pointed out, in a similar but non-supersymmetric context, in [65]. The point is that the type IIA string has a coupling $\exp\left(\mathrm{i} \int_\Sigma B\right)$. When we insert this string into our string theory path integral, we should include this coupling in the computation of the action. Moreover, in the supergravity partition function we should remember to sum over flat $B$-fields. Adding a closed $B$-field does not affect the supergravity equations of motion, but different closed $B$-fields can be gauge inequivalent, and should be summed/integrated over. This is a key point.

In the present situation, with boundary conditions fixed at infinity, we should sum over $B$-fields in spacetime that are zero at infinity, modulo shifts $B \to B + \mathrm{d}\Lambda$, where $\Lambda$ is also zero at infinity. This means that physically distinct $B$-fields, with fixed boundary

[1]For $Y_7 = S^7$, or $S^7/\mathbb{Z}_4$ as appropriate for a Taub-Bolt-AdS solution, the string can be at any point on the $\mathbb{CP}^3$ base. More generally, it will sit at a point in the IIA base $M_6$ in such a way that the M-theory circle fibre above it is calibrated and hence BPS.

condition at infinity, are measured by $H^2_{\rm cpt}(\mathcal{M}^{(4)})$. In fact including large gauge transformations this becomes $H^2_{\rm cpt}(\mathcal{M}^{(4)}, U(1))$. The key point is that for the Taub-NUT-AdS spacetime this group is zero, so there are no flat $B$-fields to sum over. But for the Taub-Bolt-AdS spacetimes, because of the $S^2$ bolt in fact $H^2_{\rm cpt}(\mathcal{M}^{(4)}, \mathbb{R}) \cong \mathbb{R}$, and is generated by a closed two-form that integrates to 1 over the fibre of $\mathcal{M}^{(4)} = \mathcal{M}_1 = \mathcal{O}(-1) \to S^2$, and has rapid decay up the fibre. Including large gauge transformations, this means there is an $S^1$ moduli space of $B$-fields to integrate over, and the supergravity saddle point approximation for the path integral with the type IIA string inserted should be

$$\langle\, \text{string}\, \rangle_{\rm Bolt} \;=\; \int_{\vartheta=0}^{2\pi} \exp\left[-A_{\rm string} + \mathrm{i}\vartheta\right] \;=\; 0\ . \tag{G.1}$$

Here $A_{\rm string}$ is the area of the string (its action), while $\vartheta$ parametrizes the different $B$-fields integrated over the fibre. For the Taub-NUT-AdS solution, there is no such integral, and the VEV is just given by the classical area, in the large $N$ limit. On the other hand, this argument shows that the VEV of the Wilson loop in the Taub-Bolt-AdS backgrounds is *identically zero*.

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